Information about the Proficiency Exam on Foundations of Computer Science

Proficiency exams are intended for students who who wish to get transfer credit for courses taken at other schools. The proficiency exam on *Foundations of Computer Science* covers the course material of CSE 215 and includes the following topics:

Set Theory
Relations
Functions
Recursion and Functional Programming
Elementary Logic
Mathematical Induction

Sample problems, with solutions, for each topic are included below. Most of these topics are discussed in textbooks on the application of discrete mathematics to computer science. The following textbook has been used at Stony Brook for CSE 215:

> Susanna S. Epp, *Discrete Mathematics with Applications.* Third edition, Brooks/Cole Publishing Company 2004.

Consult this or other textbooks for a detailed exposition and further exercises.

Set Theory

1. Let S be the set $\{\{\}, \{1,2,3\}, \{\{4,5\},6\}\}$. Indicate for each of the following statements whether it is true or false.

   (a) $\{\}$ is a subset of S.

   Answer: True (the empty set is a subset of any set)

   (b) $\{\}$ is an element of S.

   Answer: True (the empty set is one of the three elements of S)

   (c) $\{1,2,3\}$ is a subset of S.

   Answer: False (the set $\{1,2,3\}$ is an element of S, not a subset)

   (d) The power set of S has 9 elements.

   Answer: False (it has $2^3 = 8$ elements)

2. Refute the following statements by giving counterexamples.

   (a) If $A \cup B = B \cup C$, then $A = C$.

   Solution: Take $A = \{0\}$, $B = \{0,1\}$, and $C = \{1\}$. Then $A \cup B = \{0,1\} = B \cup C$, but $A \neq C$.

   (a) $(A \cup B) \cap C = A \cup (B \cap C)$

   Solution: Take $A = \{1\}$ and $C = \{\}$. Then

   $$(A \cup B) \cap C = \{\} \neq \{1\} = A \cup (B \cap C).$$

3. Consider the sets $A = \{1,2,4,7,8\}$, $B = \{1,5,7,9\}$, and $C = \{3,7,8,9\}$, with universal set $U = \{1,2,3,...,10\}$. Express each of the following sets in terms of A, B, and C via union, intersection, set difference, and complement.

   (a) $\{7\}$

   Solution: $A \cap B \cap C = \{7\}$

   (b) $\{6,10\}$

   Solution: $(A \cup B \cup C)^c$

   (c) $\{8,9\}$

   Solution: $((A \cap C) \setminus B)) \cup ((B \cap C) \setminus (A \cap B \cap C))$

Relations

4. Let R be the binary relation $\{(a, b), (b, c), (c, a)\}$ on the set $\{a, b, c\}$; S the binary relation $\{(x, y) : xy \geq 0\}$ on the integers; and T the binary relation $\{(m, n) : m+n \text{ is odd}\}$ on the integers. Indicate which properties the relations satisfy.

	R	S	T
symmetric			
transitive			
reflexive			
irreflexive			

Solution: R is only irreflexive; S is symmetric, transitive, and reflexive; and T is symmetric and irreflexive.

5. A partial order is a reflexive, transitive, antisymmetric relation. Define partial orders with the following properties on the set of natural numbers (i.e., the set of nonnegative integers).

   (a) A partial order that has one maximal element and no minimal element.

   Solution: The relation $\geq$ satisfies the required properties. The number 0 is maximal in this order, as there is no natural number k distinct from 0 for which $0 \geq k$. In addition, there is no minimal element in this order, as for each natural number k, there exists another natural number j with $j \geq k$ (e.g., $j = k + 1$).

   (b) A partial order that has infinitely many maximal elements and one minimal element.

   Solution: The relation $\{(0, n) : n \text{ a natural number}\} \cup \{(n, n) : n \text{ a natural number}\}$ has the desired properties. It is a partial order in which 0 is a minimal element and all other elements are maximal.

Functions

6. Indicate whether the following functions are one-to-one or onto (or both), where $\mathbf{Z}$ denotes the set of integers and $\mathbf{N}$ the set of nonnegative integers.

   (a) $f : \mathbf{N} \to \mathbf{N}$, where $f(x) = x^2$

   Solution: The function is one-to-one but not onto.

   (b) $f : \mathbf{Z} \to \mathbf{Z}$, where $f(x) = x^2$

   Solution: The function is neither one-to-one nor onto. For instance, $f(1) = 1 = f(-1)$ so that f is not one-to-one. Since $f(x) \geq 0$ for all integers x, the function is not onto either.

7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Determine whether the following statements are true or false. Correspondingly, give a proof or counterexample.

   (a) If f is one-to-one, then the composition $g \circ f : X \to Z$ is also one-to-one.

   Solution: The statement is false. For instance, let f be the identity function on the set $\{0, 1\}$ and g be a function with $g(0) = g(1)$. Then f is one-to-one, but $g \circ f$ is not, as $g(f(0)) = g(0) = g(1) = g(f(1))$ and $0 \neq 1$.

   (b) If $g \circ f : X \to Z$ is one-to-one, then f is one-to-one.

   Solution: We prove this statement by contradiction. Suppose $g \circ f$ is one-to-one, but f is not. Then there are elements a and b in X, such that $a \neq b$ and $f(a) = f(b)$. But then we also have $g(f(a)) = g(f(b))$, which contradicts the assumption that $g \circ f$ is one-to-one.

Recursion and Functional Programming

8. Give a recursive definition of a function *sum* that maps a pair (S, n), where n is a positive integer and S is an integer sequence $S = (a_1, \ldots, a_n)$ of length n, to the sum $a_1 + \ldots + a_n$. You may use auxiliary functions $first$ and $rest$, defined on non-empty sequences with $first(S) = a_1$ and $rest(S) = (a_2, \ldots, a_n)$.

   Solution:

$$sum(S,n) = \begin{cases} first(S) & \text{if } n = 1 \\ first(S) + sum(rest(S), n-1) & \text{if } n > 1 \end{cases}$$

9. Give a recursive definition of a function *sumpos* that takes as argument a list of integers L and returns the sum of all *positive* integers in L. For example, `sumpos([2,-1,3,0,-4,7])=12`. Use the basic list functions *head* and *tail*.

   Solution:

$$sumpos(L) = \begin{cases} 0 & \text{if } L = nil \\ head(L) + sumpos(tail(L)) & \text{if } L \neq nil \text{ and } head(L) > 0 \\ sumpos(tail(L)) & \text{otherwise} \end{cases}$$

Propositional Logic

10. Determine for each of the following propositional formulas whether it is a tautology, a contradiction, or neither.

    (a) $p \to \neg p$

    Solution: This formula is neither a tautology nor a contradiction, as it evaluates to *true* if p is *false*, and to *false* if p is *true*.

    (b) $((p \to q) \to p) \to p$

    Solution: This formula is a tautology. It evaluates to *true* for all possible truth assignments to the variables p and q, as can be seen by constructing a corresponding truth table.

11. Bill, Sue and Fred are suspected of civil disobedience. They testify under oath as follows.

    Bill: *Sue is guilty and Fred is innocent.*
    Sue: *If Bill is guilty, then so is Fred.*
    Fred: *I am innocent, but at least one of the others is guilty.*

    Assume that a person is either innocent or guilty, but not both.

    (a) Assuming that everyone is innocent, who committed perjury (who lied)?

    Solution: If everyone is innocent, then Bill and Fred evidently lied, but Sue told the truth.

    (b) Assuming that everyone's testimony is true, who is innocent and who is guilty?

    Solution: Suppose everyone told the truth. From Bill's testimony we can infer that Sue is guilty and Fred is innocent. But then Bill must be innocent, for otherwise Sue's statement would be false.

12. Show how the connectives $\wedge$ (conjunction), $\oplus$ (exclusive disjunction) and the constant T (true) can be used to express $\vee$ (inclusive disjunction).

    Solution: $\alpha \vee \beta$ is logically equivalent to $((\alpha \oplus T) \wedge (\beta \oplus T)) \oplus T$.

13. Determine whether $p \leftrightarrow \neg q$ is logically equivalent to $(p \vee q) \wedge (\neg p \vee \neg q)$.

    Solution: The first formula is logically equivalent to the conjunction $(p \to \neg q) \wedge (\neg q \to p)$. The latter formula is equivalent to $(\neg p \vee \neg q) \wedge (\neg\neg q \vee p)$, which by double negation and commutativity is equivalent to the second formula above.

Predicate Logic

14. Translate the following sentences into predicate logic formulas, using unary predicates U, Z, B, and S to represent, respectively, the properties of being a unicorn, a zebra, blue, or striped.

    (a) Unicorns exist.

    Solution: $\exists x U(x)$

    (b) Unicorns do not exist.

    Solution: $\neg\exists x U(x)$

    (c) All zebras are striped.

    Solution: $\forall x[Z(x) \rightarrow S(x)]$

    (d) All blue zebras are striped.

    Solution: $\forall x[Z(x) \wedge B(x) \rightarrow S(x)]$

    (e) Some striped zebras are blue.

    Solution: $\exists x[Z(x) \wedge B(x) \wedge S(x)]$

15. Indicate whether the statements below are true or false over the domain of natural numbers $\{0, 1, 2, 3, \ldots\}$.

    (a) $\forall x\, [x > 17 \vee x < 12]$

    Answer: False

    (b) $\exists x\, [x > 17 \rightarrow x < 12]$

    Answer: True

    (c) $\exists y\, \forall x\, (y > x)$

    Answer: False

    (d) $\exists y\, \forall x\, (x \geq y)$

    Answer: True

16. Let F be the set of all foxes, R the set of all rabbits, and A the set of all animals. Use the predicates $fox(X)$, $rabbit(X)$, $color(X,\, Y)$ and $eats(X,\, Y)$ and the domain A to express the *negation* of each of the following formulas as a formula beginning with a quantifier.

    (a) $\forall f \in F,\ color(f, red)$

    Solution: $(\exists a \in A)(fox(a) \wedge \neg color(a, red))$

    (b) $\exists r \in R,\ color(r, white)$

    Solution: $\forall(a \in A)(rabbit(a) \rightarrow \neg color(a, white))$

Mathematical Induction

17. Use mathematical induction to prove that for all integers $n \geq 1$,

$$\sum_{j=1}^{n}(2j-1) = 1 + 3 + \cdots + (2n-1) = n^2.$$

Solution: Let $P(n)$ be the property that $\sum_{j=1}^{n}(2j-1) = n^2$.
Basis case. If $n = 1$, then

$$\sum_{j=1}^{n}(2j-1) = \sum_{j=1}^{1}(2j-1) = (2-1) = 1^2 = n^2.$$

Thus $P(1)$ is true.
Induction step. We show that $P(n)$ implies $P(n+1)$, for all $n \geq 1$. Let k be a fixed integer with $k \geq 1$ and assume, as induction hypothesis, that $P(k)$ is true, i.e.,

$$\sum_{j=1}^{k}(2j-1) = k^2.$$

We need to show that $P(k+1)$ is true, or

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2.$$

We have,

$$\begin{aligned}\sum_{j=1}^{k+1}(2j-1) &= [\sum_{j=1}^{k}(2j-1)] + 2(k+1) - 1 \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2.\end{aligned}$$

This completes the proof.