Seeing the Forest for the Trees

One problem / danger of the textbook is the amount of detail / digressions. It is important to realize that there are no general methods for most of these problems and so tricks are important, but they include lots of cute tricks which are not very important.

So, rest assured that anything I test you on will be discussed in class, and that you are not expected to master all digressions.

Read the book with an eye to what is important so you don't get overwhelmed with detail.

Recurrence Relations

A recurrence relation is an equation which is defined in terms of itself.

$$a_n = 2a_{n-1} + a_{n-2} + 5$$

$$a_n = 2a_{n-1}^2$$

Recurrence relations are good things for many reasons

(1) Many different functions can be naturally expressed as recurrences.

$a_n = a_{n-1} + 1, \; a_1 = 1$ $\qquad a_n = n$ (polynomial)

$a_n = 2a_{n-1}, \; a_1 = 1$ $\qquad a_n = 2^{n-1}$ (exponential)

(2) Because of the close connection between recursion and induction, the time complexities of algorithms are often defined by recursion.

(3) It is often easy to find a recurrence as the solution of a counting problem.

Solving the recurrence is often harder and is the focus of the text.

Recursion <u>is</u> Mathematical Induction!

In both, we have general and boundary conditions, with the general condition breaking the problem into smaller and smaller pieces.

The <u>initial</u> or boundary conditions terminate the recursion.

As we will see, mathematical induction provides a useful tool to solve recurrences - guess a solution and prove it inductively

$$T_N = 2T_{N-1} + 1, \quad T_0 = 0$$

Prove $T_N = 2^N - 1$ by induction.

$T_0 = 2^0 - 1 = 0$ (basis)

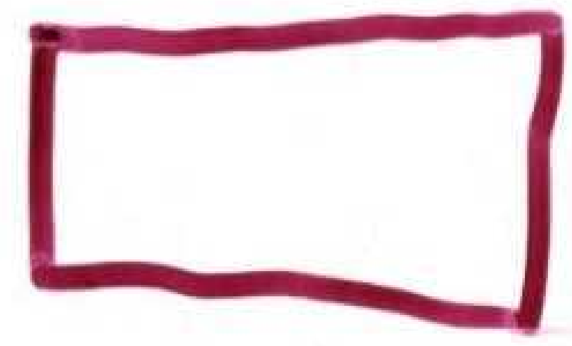
assume True for T_{N-1}

$$T_N = 2T_{N-1} + 1$$
$$= 2(2^{N-1} - 1) + 1 = 2^N - 1$$

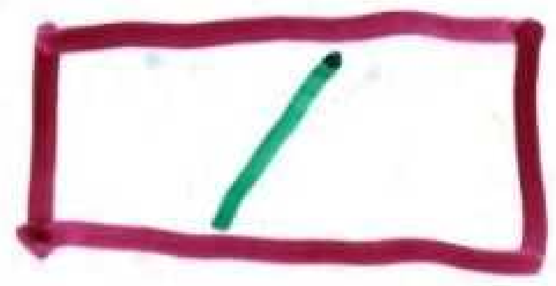
Pizza Slices in the Plane

Given N lines in the plane, what is the maximum number of regions they define?

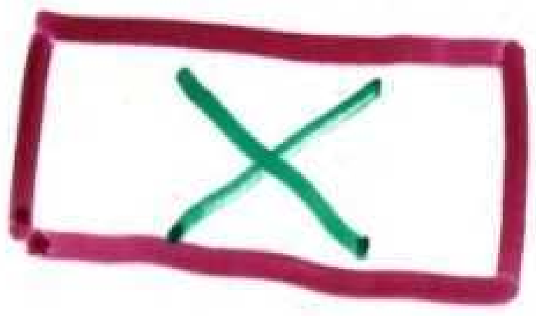
$L_0 = 1$



$L_1 = 2$

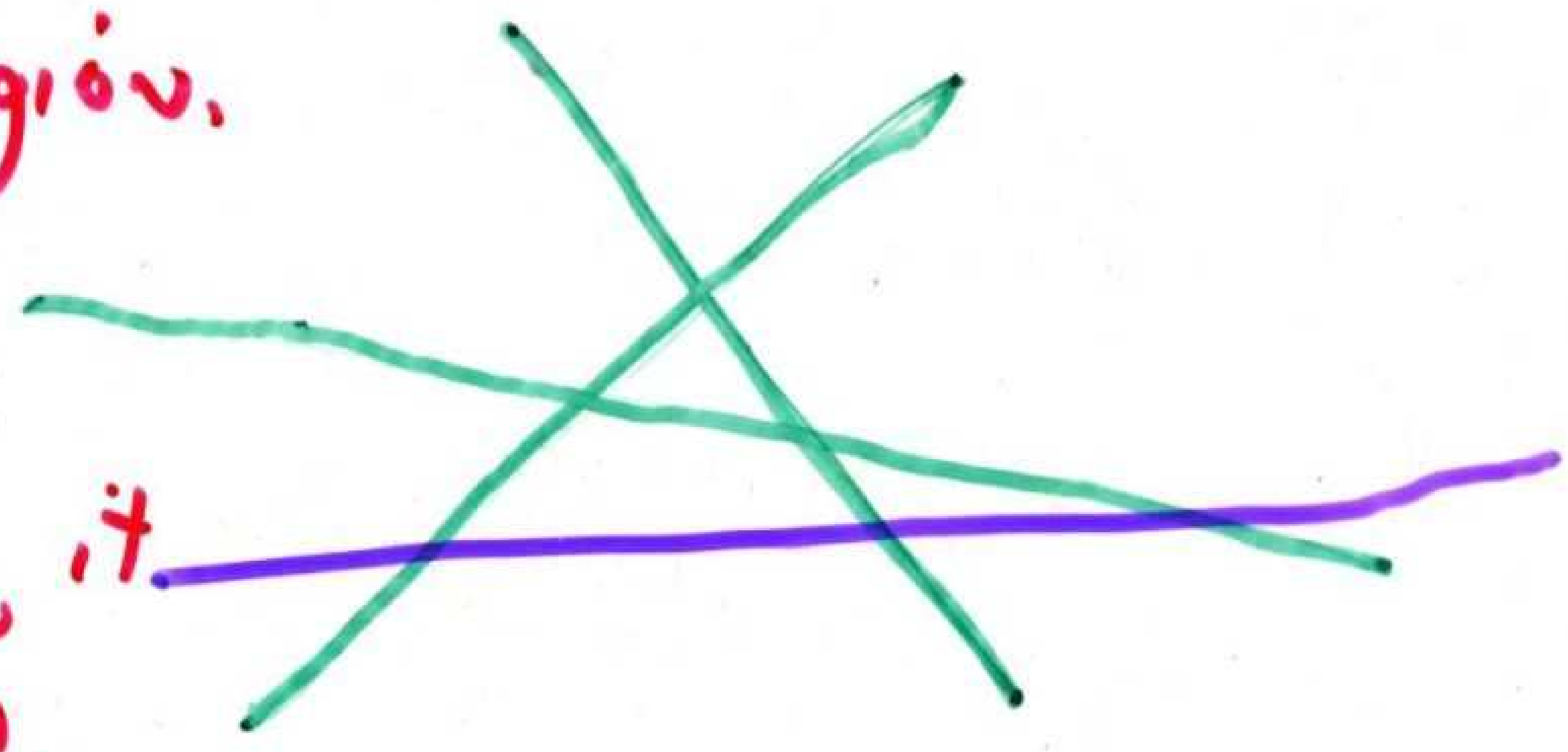


$L_2 = 4$



$L_3 = ?$

Each new line adds a region whenever it splits an old region.



Whenever the line crosses an old line, it enters a new region,

and so the $N^{\underline{th}}$ adds at most N regions, since $(N-1)$ intersections defines N intervals.

$L_0 = 1$

$$L_N = L_{N-1} + N = 1 + \sum_{i=1}^{N} i = \frac{N(N+1)}{2} + 1$$

$L(N)$	1	2	3	....	N
$+L(N)$	N	$N-1$	$N-2$	...	1
$2(L_N)$	$N+1$ ...		..		$N+1$

$= N(N+1)$

The Josephus Problem

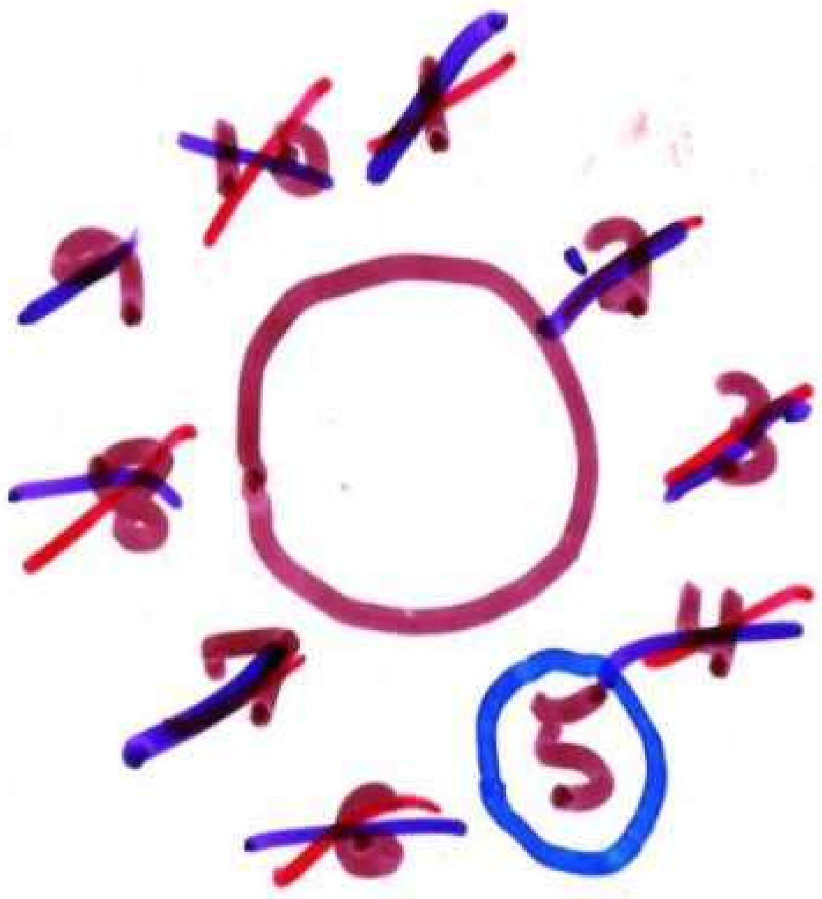
Josephus was a general on the losing side (originally of the Jewish-Roman war. With a band of 41 men who prefered suicide to capture, it was decided to form a circle and slay every third man until none were left. Josephus was not into this die with glory business, and so calculated where to stand to be the last to survive, and then went over to the Roman side as a trator....

The Josephus problem is to determine who the last survivor is in a circular permutation of N men with every k^{th} person executed.

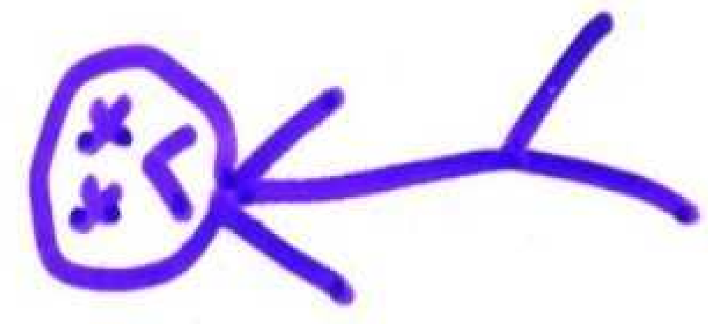
According to Rouss Ball, the problem originated in the Middle ages with an overcrowded ship of Turks and Christians, with the goal to find a way to throw the Turks overboard.

Religious bias is something of a problem in math problems - witness missionaries and cannibles!

Let us consider the special case of the Josephus problem where every second man gets slain



2, 4, 6, 8, 10,
3, 7, 1, 9, 5



With 10 men, the last survivor is 5.

Let $J(n)$ be the last survivor.
How can we calculate this quickly?

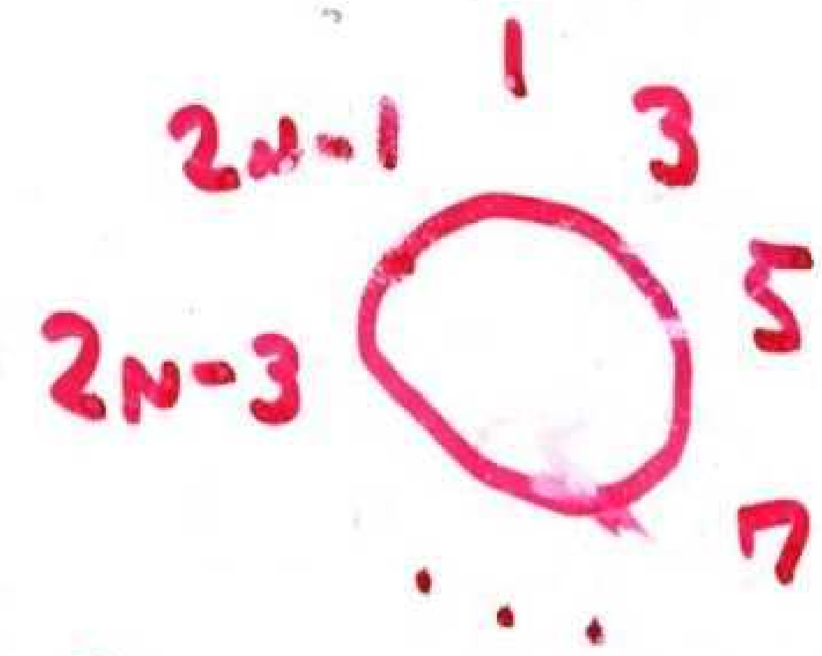
With a recurrence. $J(1) = 1$

n	1	2	3	4	5	6	7
$J(n)$	1	1	3	1	3	5	1

Our first observation is that the even numbered guys die first. If we start out with an even number of $2n$ men, the first trip through the circle eliminates all the even ones.

Whats left?

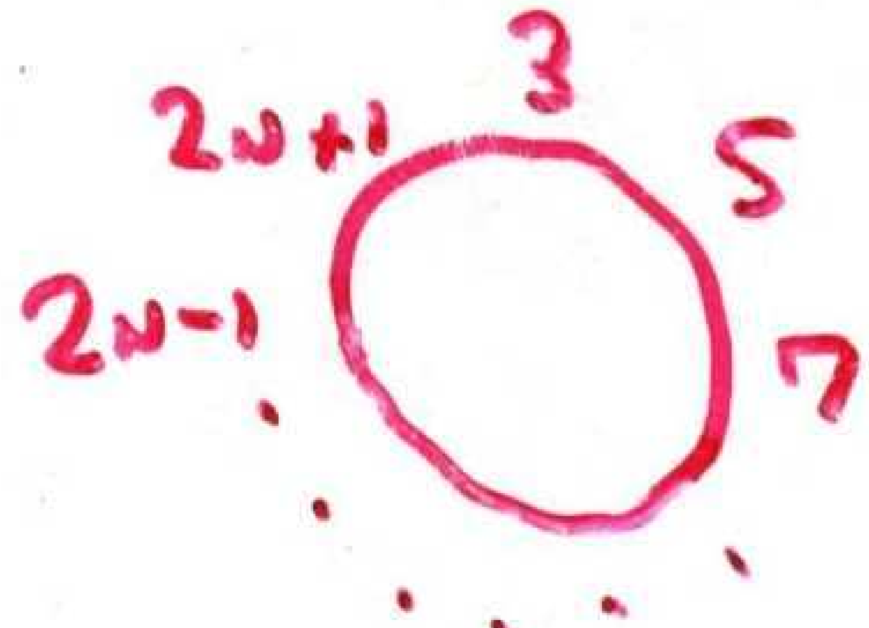
An instance of size N of odd numbered peoples, so..



$1 \to 1$
$2 \to 3$
$3 \to 5$
$\vdots$

$$J(2N) = 2J(N) - 1$$

To complete the problem, we must consider the odd case. After going around once, the first guy is the next to go, so:



$$J(2N+1) = 2J(N) + 1$$

$1 \to 3$
$2 \to 5$
$\vdots$

Together, these completely define the function:

$$J(1) = 1$$
$$J(2N) = 2J(N) - 1$$
$$J(2N+1) = 2J(N) + 1$$

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J_N	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

We need a closed form to provide an even faster way to compute $J(n)$, since we may use the Josephus algorithm to reduce enrollment in this class!

Looking at the data, it seems reasonable to conjecture $J(2^m + \ell) = 2\ell + 1$, $m \geq 0$ + $0 \leq \ell < 2^m$.

We can prove it by induction on exponent $\underline{m}$

(BASIS) $J(1) = J(2^0 + 0) = 2(0) + 1 = 1$ ✓

(n is even $\Rightarrow$ ℓ is even) $J(2^m + \ell) = 2J(2^{m-1} + \ell/2) - 1$
$= 2(\ell + 1) - 1 = \underline{2\ell + 1}$ ✓

(n is odd $\rightarrow$ ℓ is odd) $J(2^m + \ell) = 2J\left(2^{m-1} + \frac{(\ell - 1)}{2}\right) + 1$
$= 2(\ell - 1 + 1) + 1 = \underline{2\ell + 1}$ ✓

Thus if you can find the largest power of 2 less than n, you can save your skin!

$J(100) = J(64 + 36) = 2 \cdot 36 + 1 = \underline{73}$

The Repertoire Method of Solving Recurrences

Man does not live by induction alone! There are better ways to solve recurrences, in that they provide more insight.

Let us solve a generalization of the Josephus recurrence:

$$f(1) = \alpha$$
$$f(2n) = 2f(n) + \beta$$
$$f(2n+1) = 2f(n) + \gamma$$

n	$f(n)$
1	α
2	$2\alpha + \beta$
3	$2\alpha + \quad + \gamma$
4	$4\alpha + 3\beta +$
5	$4\alpha + 2\beta + \gamma$
6	$4\alpha + \beta + 2\gamma$
7	$4\alpha + \quad + 3\gamma$
⋮	

It seems clear that the solution is a function of α, β, and γ.

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

We can find the values of $A(n)$, $B(n)$, and $C(n)$ by finding three independent sets of α, β, γ and combining the specific cases.

Ex: $\alpha = 1$, $\beta = -1$, $\gamma = 1$ gives the original solution.
We seek examples which will give us simple solutions.

(1) $\alpha = 1$, $\beta = \gamma = 0$

$f(1) = 1$
$f(2n) = 2f(n)$
$f(2n+1) = 2f(n)$
$\rightarrow f(n) = 2^m$, where $n = 2^m + l$, $0 \le l \le 2^m$

(2) Another simple case is $f(n) = 1$.
To see if we can generate this function from the recurrence, we back substitute to get the α, β, γ.

$f(n) = f(2n) = 1$

$1 = \alpha$
$1 = 2 \cdot 1 + \beta$
$1 = 2 \cdot 1 + \gamma$

$\alpha = 1$
$\beta = -1$
$\gamma = -1$

(3) A third simple case is $f(n) = n$

$$1 = \alpha \qquad \alpha = 1$$

$$2n = 2n + \beta \qquad \beta = 0$$

$$2n+1 = 2n + \gamma \qquad \gamma = 1$$

What do we now know? That if

$$f(n) = A(n)\,\alpha + B(n)\,\beta + C(n)\,\gamma$$

is a solution, then

$$1 \cdot A(n) + 0 \cdot B(n) + 0 \cdot C(n) = 2^m$$

$$1 \cdot A(n) + -1 \cdot B(n) + -1 \cdot C(n) = 1$$

$$1 \cdot A(n) + 0 \cdot B(n) + 1 \cdot C(n) = n$$

so:

$$A(n) = 2^m$$

$$C(n) = n - 2^m = l$$

$$B(n) = 2^m - C(n) - 1$$

$$= 2^m - l - 1$$

$$\therefore\ f(n) = \alpha 2^m + (2^m - l - 1)\beta + \gamma l.$$

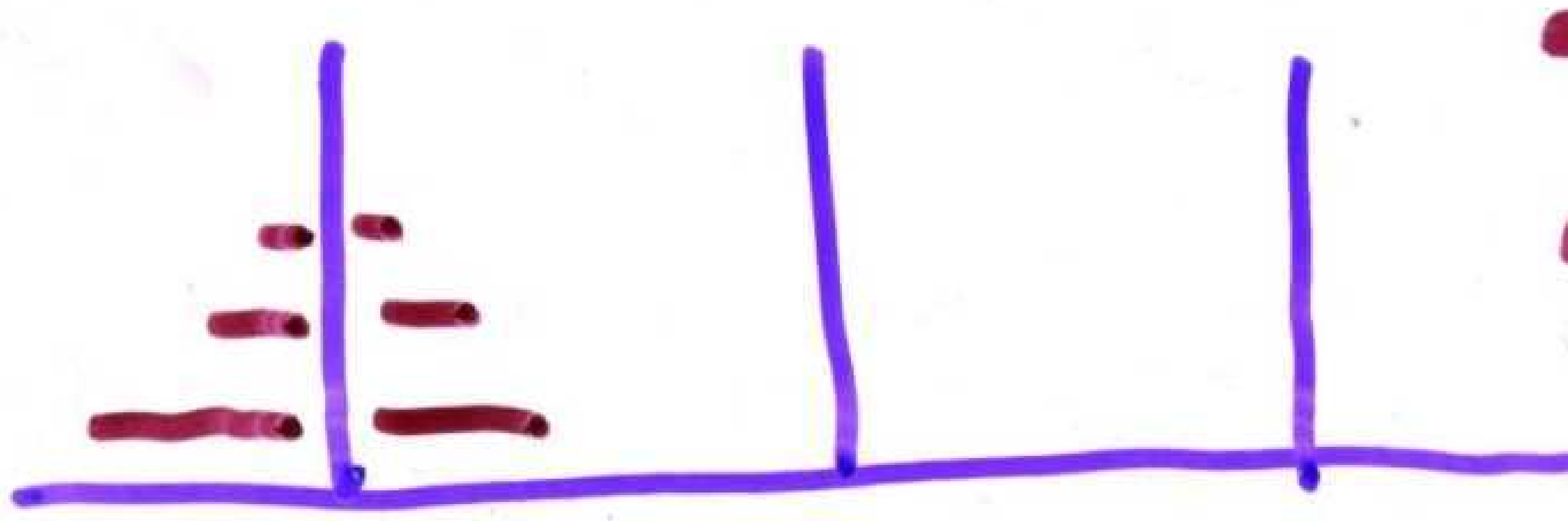
The repertoire method, like most ways of solving recurrences, is a black art. It will give you the solution when you can guess the form of the solution, for example with linear recurrences.

It is a *very* good practice to prove by induction any results obtained by this method, to guard against errors and non-independent solutions.

Later this semester, we will see how generating functions can be used to provide less magical ways to solve recurrences.

The Tower of Hanoi

As you all know, the tower consists of stacked disks in decreasing order on a peg, with the goal to move them to another peg using a third as intermediate. At no point can a large disc sit on a small one.



What is $H(n)$, the number of moves to solution for n disks?

$$H(1) = 1$$

By moving the top $n-1$ to the other peg, move the bottom, and move the $n-1$ back,

$$H(n) \leq 2H(n-1) + 1$$

Since before moving the largest disk we must move the others to third peg, and after moving it for the last time we must move them back, this gives a matching lower bound

$$H(n) \geq 2H(n-1) + 1$$

What about $k > 3$ pegs?