Harmonic Numbers

When we were working with sums the quantity $\sum_{i=1}^{N} 1/i = H_N$ occurs often enough to justify its own name. We saw that it diverges, but what is it close to?

$$\underbrace{\frac{1}{1}}_{} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} \cdots\cdots \frac{1}{15}}_{}$$

If we group the fractions such that the i^{th} group contains 2^{i-1} fractions, the fractions in the i^{th} group are:

$$\underbrace{\frac{1}{2^{i-1}}}_{\text{MAX}} \cdots \underbrace{\frac{1}{2^{i}-1}}_{\text{MIN}}$$

Thus the sum of the i^{th} group is AT LEAST

$$2^{i-1} \cdot \frac{1}{2^{i}-1} \approx 1/2$$

and at most $2^{i-1}/2^{i-1} = 1$, so

$$\frac{\lfloor \lg N \rfloor + 1}{2} \leq H_N \leq \lfloor \lg N \rfloor + 1$$

Using higher order harmonics we can get an even tighter bound on H_N:

$$\ln\left(\frac{k}{k-1}\right) = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} \cdots$$

Thus $\ln k - \ln(k-1) = \sum_{N=1}^{\infty} \frac{1}{Nk^N}$

and..

$$\ln k - \ln(k-1) + \ln(k-1) - \ln(k-2) = \sum_{N=1}^{\infty} \left(\frac{1}{Nk^N} + \frac{1}{N(k-1)^N} \right)$$

And after telescoping:

$$\ln N - \ln 1 = \sum_{k=2}^{N} \left(\frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} \cdots \right)$$

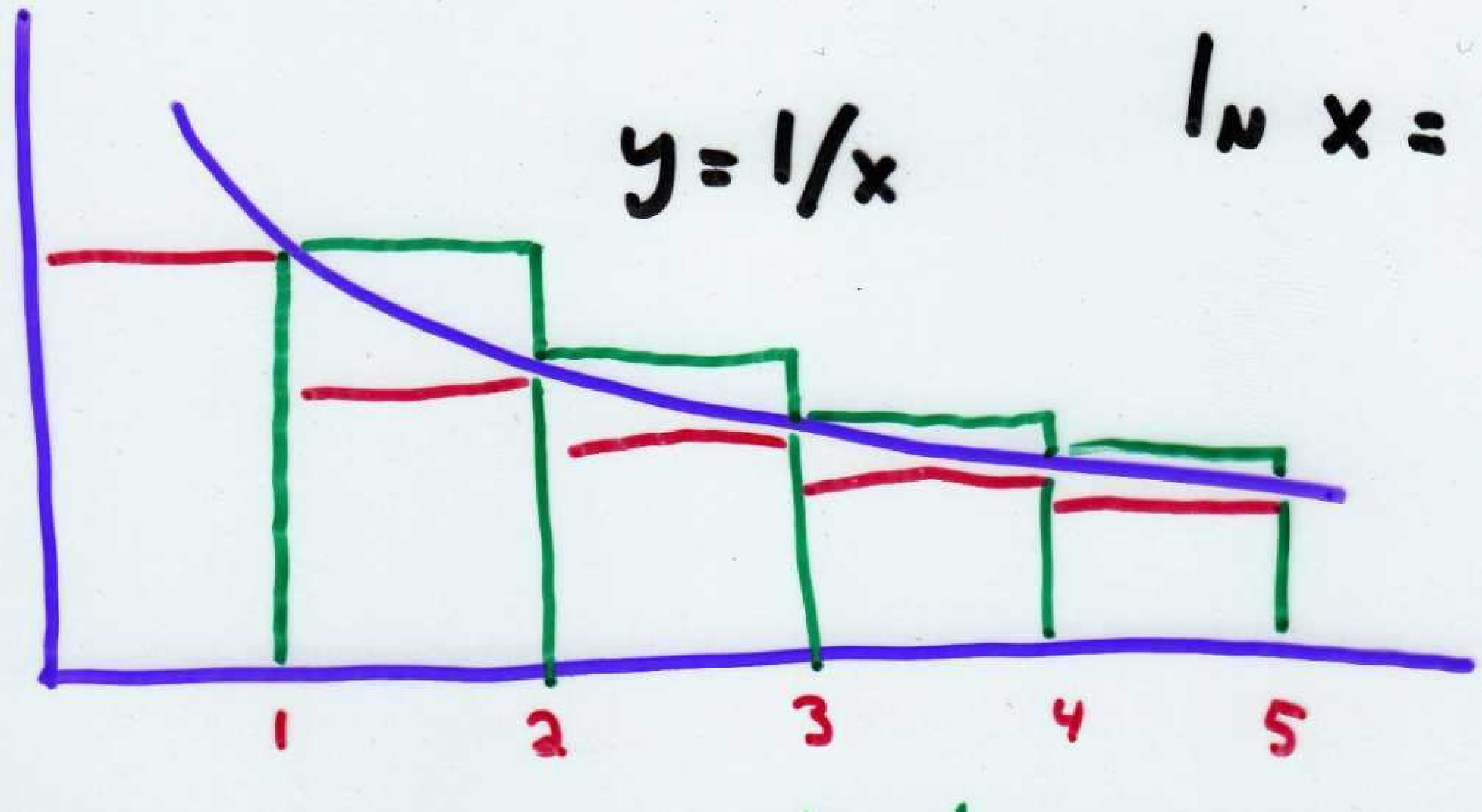
$$= H_N - 1 + \underbrace{\sum_{i=2}^{\infty} \frac{1}{i} \left(H_N^{(i)} - 1 \right)}$$

approaches a constant as $N \to \infty$, so:

$$\lim_{N\to\infty} (H_N - \ln N) = \gamma = 0{,}577216649$$

"Euler's Constant"

Tighter bounds follow from Calculus:



$$\ln x = \int_1^x 1/u \, du$$

The green rectangles over count $\ln x$, so

$$H_N \geq \ln N$$

The red rectangles -1 under count $\ln x$, so,

$$H_N - 1 < \ln N$$

So, $\ln N < H_N < \ln N + 1$, which is a very tight bound!

For all $k < -1$, $\sum_{i=1}^{\infty} i^k$ converges.

Let $H^{(k)}_{(N)} = \sum_{i=1}^{N} 1/i^k$ be the k^th^ order harmonic number.

The Riemann Zeta Function:

$$\zeta(r) = \sum_{k \geq 1} 1/k^r$$

The Fibonacci Numbers

The simpliest recurrence imaginable is $F_N = F_{N-1} + F_{N-2}$ and thus one would suspect it occurs in lots of different contexts.

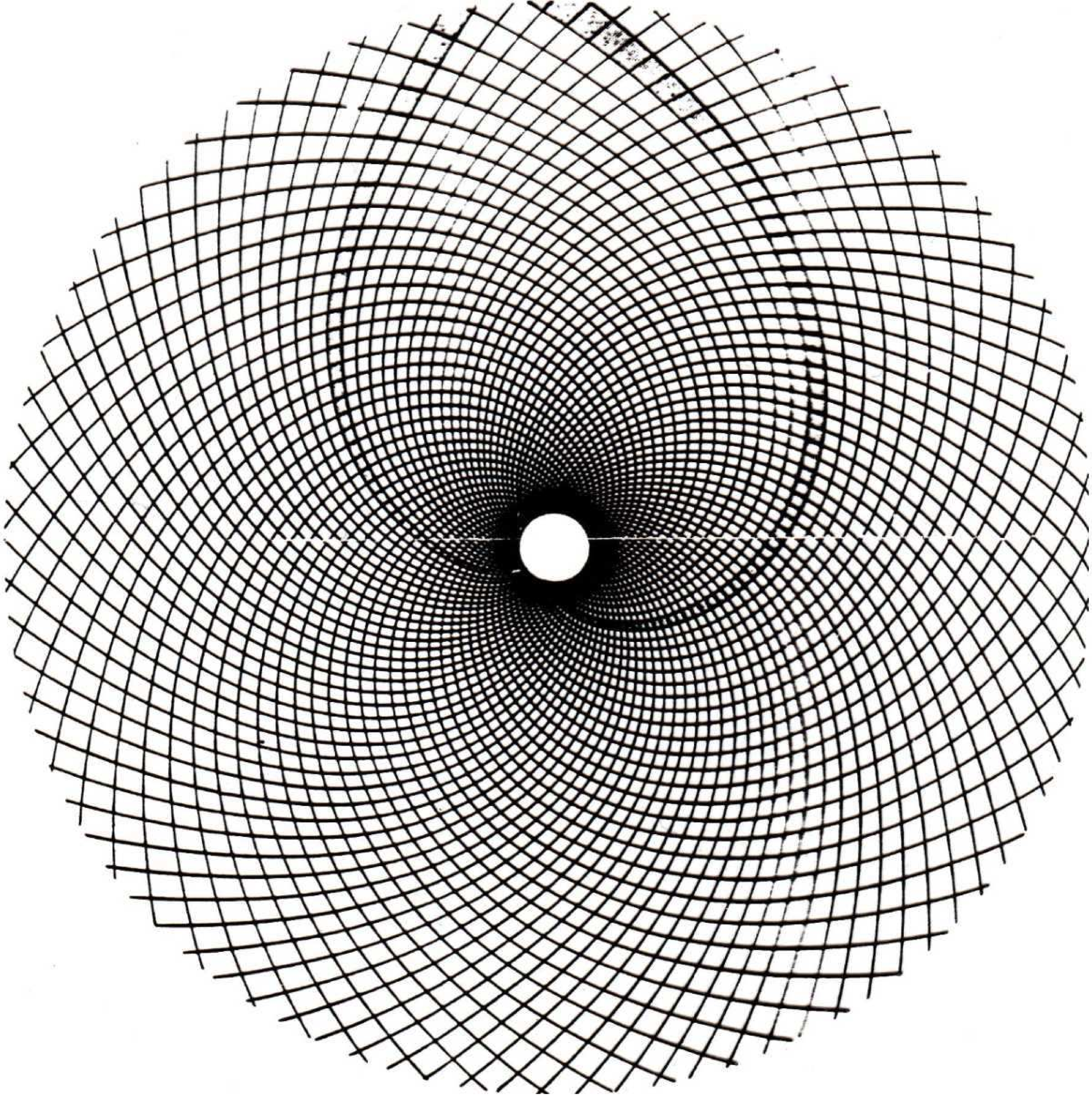
With $F_0 = 0$ and $F_1 = 1$ these describe the Fibonacci numbers, and they appear more magicly than John 3:16 signs in front of TV cameras at the World Series!

Ex: Petals in Flowers tend to be counted by Fibonacci number, so "She Loves Me, She Loves me not" has an optimal strategy

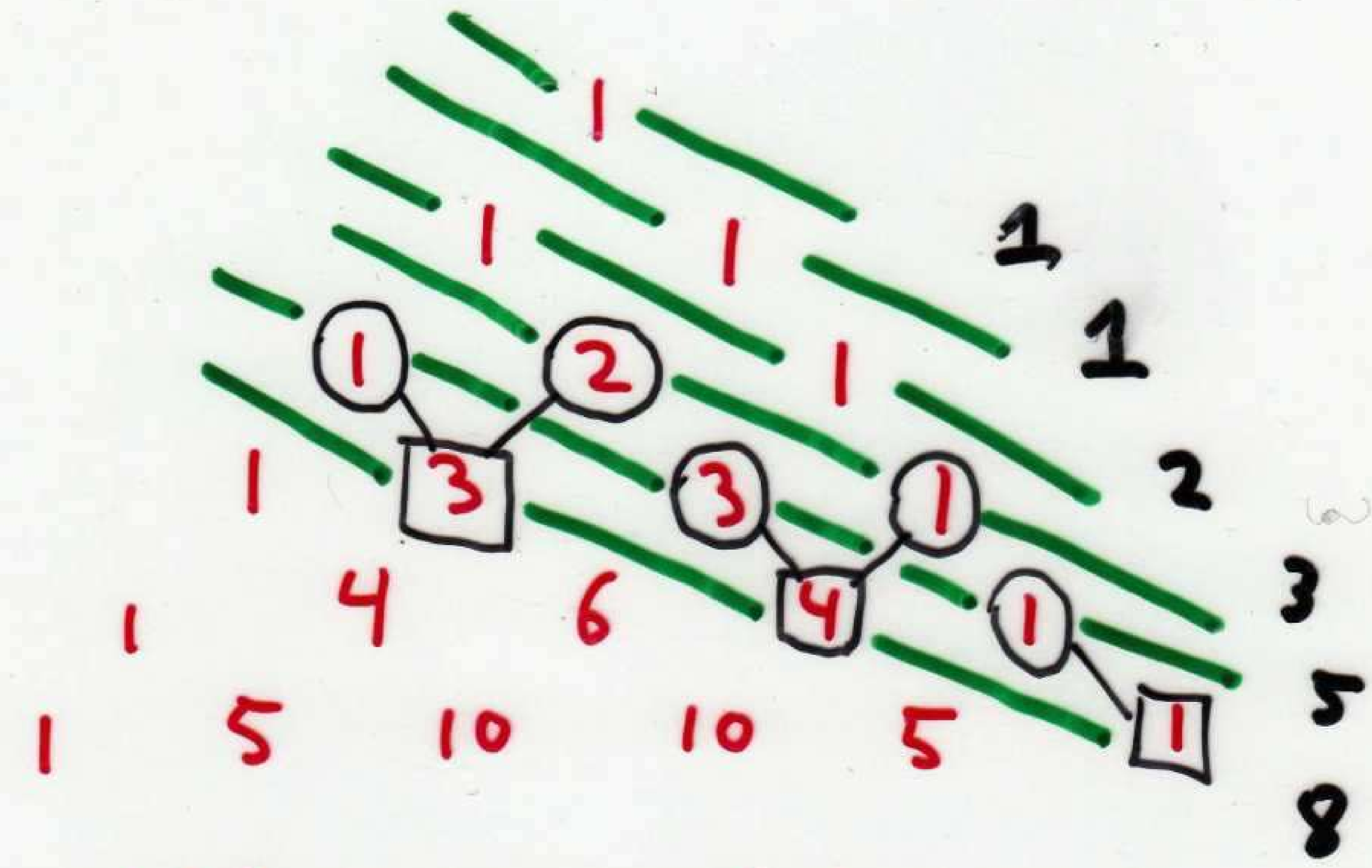
Ex: Fibonacci search is sometimes more appropriate than binary search.

N	0	1	2	3	4	5	6	7	8	9	10
F_N	0	1	1	2	3	5	8	13	21	34	55

Ex: There are lots of amazing identities about them – enough to Fill their own Journal!



Fibonacci Numbers in Pascals' Triangle



$$F_6 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$$

This is most easily seen by noting that two succesive diagonals sum up to the third!

Most identities with Fibonacci numbers can be easily proven by induction or perturbation:

$$S_N = \sum_{i=1}^{N} F_i$$

$$F_N + S_{N-1} = \sum_{i=1}^{N} F_i = \sum_{i=1}^{N} (F_{i-1} + F_{i-2})$$

$$= \sum_{i=1}^{N} F_{i-1} + \sum_{i=1}^{N} F_{i-2}$$

$$= (S_{N-1} + \cancel{F_0}^{\,0}) + (S_{N-2} + \cancel{F_0}^{\,0} + F_{-1})$$

$F_{-1} = F_1 - F_0$

$$F_N + \cancel{S_{N-1}} = \cancel{S_{N-1}} + S_{N-2} + 1$$

$$\boxed{S_N = F_{N+2} - 1}$$

$1 + 1 + 2 + 3 + 5 = 13 - 1$

Proof by induction:

$$S_{N+1} = F_{N+1} + S_N = F_{N+1} + F_{N+2} - 1$$

$$= F_{N+3} - 1$$

■

Cassini's Identity

$$F_{N+1} F_{N-1} - F_N^2 = (-1)^N$$

Assume it holds up to N, so $F_{N-1} = F_{N+1} - F_N$

and:

$$F_{N+1}(F_{N+1} - F_N) - F_N^2 = (-1)^N$$

$$F_{N+1}^2 - F_N \underbrace{(F_{N+1} + F_N)}_{F_{N+2}} = (-1)^N$$

$$-F_{N+2} F_N + F_{N+1}^2 = (-1)^N \quad \therefore$$

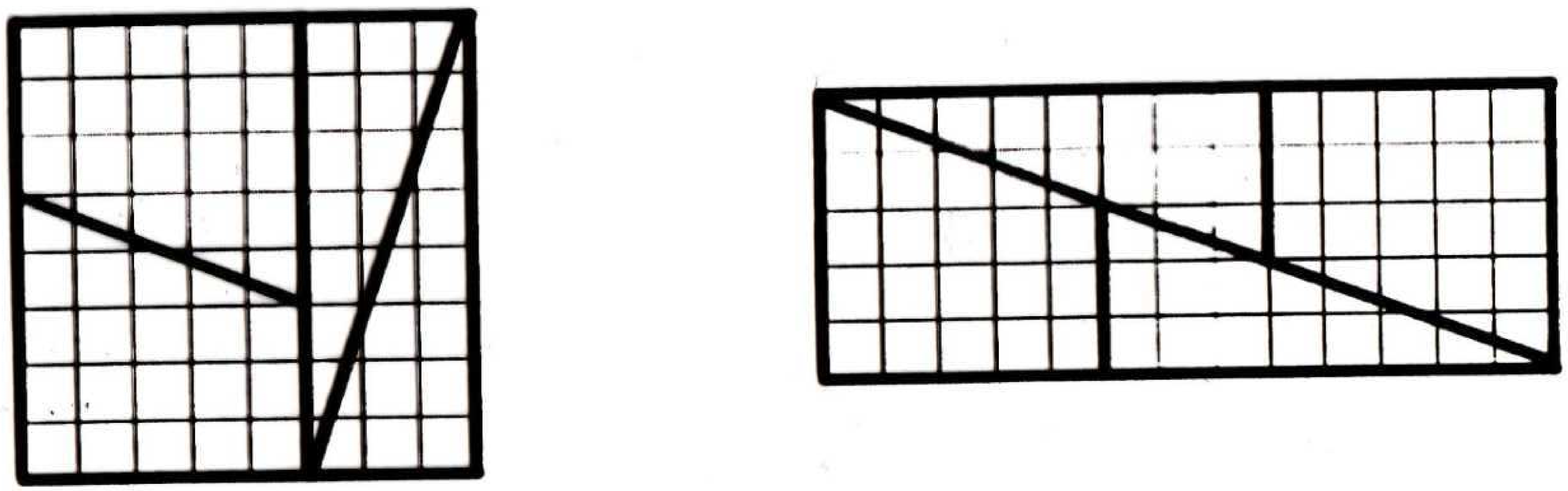
$$F_{N+2} F_N - F_{N+1}^2 = (-1)^{N+1}$$

Other Identities:

$$F_{N+k} = F_k F_{N+1} + F_{k-1} F_N$$

$$\gcd(F_n, F_N) = F_{\gcd(m,N)}$$

Cassini's identity is the basis of a geometrical paradox that was one Lewis Carroll's favorite puzzles [54], [258], [298]. The idea is to take a chess-board and cut it into four pieces as shown here, then to reassemble the pieces into a rectangle:



Presto: The original area of $8 \times 8 = 64$ squares has been rearranged to yield $5 \times 13 = 65$ squares! A similar construction dissects any $F_n \times F_n$ square into four pieces, using F_{n+1}, F_n, F_{n-1}, and F_{n-2} as dimensions wherever the illustration has 13, 8, 5, and 3 respectively. The result is an $F_{n-1} \times F_{n+1}$ rectangle; by (6.103), one square has therefore been gained or lost, depending on wheth

The paradox is explained because . . . well, magic tricks aren't supposed to be explained.

A Closed Form for the Fibonnaci Numbers

This is an excellent example of how to solve recurrences with generating functions

$$F_N = F_{N-1} + F_{N-2}$$

We can construct a generating function whose coefficients are the Fibonnaci numbers

$$F(z) = F_0 + F_1 z + F_2 z^2 \cdots = \sum_{N \geq 0} F_N z^N$$

Since $F_N z^N = z F_{N-1} z^{N-1} + z^2 F_{N-2} z^{N-2}$, by replicating and shifting our sequence we can cancel most terms!

$$\begin{aligned}
F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots \\
-zF(z) &= \phantom{F_0 + {}} F_0 z + F_1 z^2 + F_2 z^3 + \cdots \\
-z^2 F(z) &= \phantom{F_0 + F_1 z + {}} F_0 z^2 + F_1 z^3 \cdots
\end{aligned}$$

$$F(z)\left[1 - z - z^2\right] = z$$

Nothing Up My Sleeve....

Two steps in finding a closed-form for the Fibonacci numbers might seem particularly mysterious. Here are alternate arguments you might prefer:

Why is $F(z) = z/(1-z-z^2)$?

$$F(z) = \sum_{N \geq 0} F_N z^N = \sum_{N \geq 0} (F_{N-1} + F_{N-2}) z^N$$

$$= (F_{-1} + F_0 z + F_1 z^2 + F_2 z^3 + \dots) + (F_{-2} + F_{-1} z + F_0 z^2 + F_1 z^3 + F_2 z^4 + \dots)$$

$$= F_{-1} + z \sum F_N z^N + F_{-2} + F_{-1} z + z^2 \sum F_N z^N$$

$$= F_{-1} + F_{-2} + F_{-1} z + z F(z) + z^2 F(z)$$

$F_{-1} = F_1 - F_0 = 1 \qquad F_{-2} = F_0 - F_{-1} = -1$

$F(z) = \cancel{1-1} + z + zF(z) + z^2 F(z) \qquad F(z) = z/(1-z-z^2)$

Why is $\dfrac{1}{1-az} = 1 + az + a^2z^2 + a^3z^3 + \dots$?

Many of you are discontent when I use the identity for the sum of a geometric series, because you are worried about convergence.

But what is

$$(1 + az + a^2z^2 + a^3z^3 + \dots\dots)(1-az)$$

$$= (1-az) + (az - a^2z^2) + (a^2z^2 - a^3z^3) + \dots\dots$$

$$= 1 + (-az + az) + (-a^2z^2 + a^2z^2) + (-a^3z^3 + a^3z^3) + \dots$$

$$= \underline{1}$$

Every other term is cancelled, except 1.

$$1 = (1 + az + a^2z^2 + \dots)(1-az) \rightarrow$$

$$\frac{1}{1-az} = 1 + az + a^2z^2 + a^3z^3 + \dots\dots$$

unless $az = 1$, but we won't solve for z so don't worry about it!

Thus: $$F(z) = \frac{z}{1-z-z^2}$$

If we can find a representation for this generating function as an infinite series, we are in business.

Ex: $$\frac{1}{1-az} = 1 + az + a^2z^2 + a^3z^3 \ldots$$

In fact, we can find constants so that:

$$\frac{z}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

How can we find this decomposition into partial fractions? It is equivalent to:

$$z = A(1-\beta z) + B(1-\alpha z)$$
$$1-z-z^2 = (1-\alpha z)(1-\beta z)$$

Once we find the denominators, we will automatically get values for A + B, although they do not necessarily have to be the constants we crave.

How do we obtain α & β ?

$$1 - z - z^2 = (1-\alpha z)(1-\beta z) = \alpha\beta z^2 - (\alpha+\beta)z + 1$$

$$\alpha\beta = -1$$

$$\alpha + \beta = 1 \Rightarrow \beta = 1-\alpha \Rightarrow \boxed{\alpha(1-\alpha) = -1}$$

Use the Quadratic formula: $\alpha^2 - \alpha - 1 = 0$

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$$\beta = 1 - \alpha = \frac{2}{2} - \left(\frac{1 \pm \sqrt{5}}{2}\right) = \frac{1 \mp \sqrt{5}}{2}$$

$$\boxed{\alpha = \frac{1+\sqrt{5}}{2} \qquad \beta = \frac{1-\sqrt{5}}{2}}$$

The quantity $\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is called the Golden ratio and pops up in lots of interesting places:

- data structures
- continued fractions
- Greek architecture
- human anatomy

So, $$\frac{z}{1-z-z^2} = \frac{A}{1-\left(\frac{1+\sqrt{5}}{2}\right)z} + \frac{B}{1-\left(\frac{1-\sqrt{5}}{2}\right)z}$$

and, $$z = A\left(1-\left(\frac{1-\sqrt{5}}{2}\right)z\right) + B\left(1-\left(\frac{1+\sqrt{5}}{2}\right)z\right)$$

solving gives:

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$\bar{\phi} = \frac{1-\sqrt{5}}{2}$$

$$A = \frac{1}{\sqrt{5}}, \quad B = \frac{-1}{\sqrt{5}},$$

so: $$F(z) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\phi z} - \frac{1}{1-\bar{\phi} z}\right]$$

$$= \frac{1}{\sqrt{5}}\left((1-1) + (\phi-\bar{\phi})z + (\phi^2-\bar{\phi}^2)z^2 \ldots\right.$$

$$\boxed{F_N = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N\right]}$$

<u>Amazing</u>: The $\sqrt{5}$'s go away and leave us with integers.

<u>Amazing</u>! The first term provides almost all of F_N.

$$F_N = \left\lfloor \frac{\phi^N}{\sqrt{5}} + \frac{1}{2} \right\rfloor$$

The series grows exponentially

The Zeckendorf Number System

We can represent integers uniquely in the Zeckendorf or Fibonacci number systems:

$$\ldots\ldots F_7 \; F_6 \; F_5 \; F_4 \; F_3 \; F_2 \; F_1 =$$

$$\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 21 & 13 & 8 & 5 & 3 & 2 & 1 \end{array}$$

By using a greedy algorithm, we can convert any integer to a Zeckendorf Representation:

$N = 28$

34	28	21	F_8
8	7	5	F_5
	2		F_3

Any F_k appears at most once in the greedy expansion:

$$F_k \le N < F_{k+1} \Rightarrow F_k - F_k \le N - F_k < F_{k+1} - F_k$$

$$\Rightarrow 0 \le N - F_k < \underline{F_{k-1}}$$

Since $F_N = F_{N-1} + F_{N-2}$ there is no reason why there will be two consecutive 1's.

Thus we have a "binary" number system which is not a Radix system

Each binary string with no two adjacent 1's is a unique integer.

1	2	3	4	5	6	7	
1	10	100	101	1000	1001	1010	...

How do we add in this system??

The last two bits are either 00, 10, 01

They become 01, 11, 10. Now it is binary, but might have two adjacent 1's. Since $F_N = F_{N-1} + F_{N-2}$, these can be replaced by a one in the next position. This rippling continues until no two ones are consecutive.

$$
\begin{array}{r}
010101 \\
+\ 1 \\
\hline
100000 \\
13\ 8\ 5\ 3\ 2\ 1
\end{array}
$$

How many permutations of N things have k ascents, or increasing neighbors?

A permutation $p = \pi_1 \pi_2 \cdots \pi_N$ has an ascent at j when $\pi_j < \pi_{j+1}$.

$N = 4$, $k = 2$

4 1 2 3   1 2 4 3   1 3 2 4

3 1 2 4   1 3 4 2   1 4 2 3

2 1 3 4   2 3 4 1   2 3 1 4

2 4 1 3

3 4 1 2

hence: $\left\langle {4 \atop 2} \right\rangle = 11$

Consider Special Cases

$k = 0$ The only way to have 0 ascents is the reverse permutation $N, N-1, \dots 1$

$$\left\langle {N \atop 0} \right\rangle = 1$$

$k = N-1$ The only way to have all $N-1$ pairs ascending is the sorted order:

$$\left\langle {N \atop N-1} \right\rangle = 1$$

In general, a permutation of N things has $N-1$ adjacent pairs, with k ascents and $N-k-1$ decents. If we reverse the permutation, all ascents become decents, so..

$$\left\langle {N \atop k} \right\rangle = \left\langle {N \atop N-k-1} \right\rangle$$

What is the General Case?

Suppose we insert the largest element somewhere in a permutation of the first $N-1$?

Either it splits an ascent: $12 \rightarrow 1N2$ or goes first $12 \rightarrow N12$

Or a decent: $21 \rightarrow 2N1$ or goes last: $12 \rightarrow 12N$

The first two leave the total unchanged, the second two add 1, so

$$\left\langle {N \atop k} \right\rangle = (k+1) \left\langle {N-1 \atop k} \right\rangle + (N-k) \left\langle {N-1 \atop k-1} \right\rangle$$

$\left\langle {N \atop 0} \right\rangle = 1$, $N \geq 0$ $\quad \left\langle {N \atop N} \right\rangle = 0$, $N > 0$

These are the Eulerian Numbers.

Permutations and Inversion Vectors

Each descent in a permutation is an inversion, or a pair of elements out of order.

A sorted permutation contains 0 inversions, and hence 0 descents. $1\ 2\ 3 \cdots N$

A reversed permutation contains $\binom{N}{2}$ inversions and hence $N-1$ descents. $N\ \ N-1 \ \cdots \ 1$

Thus the number of inversions (or the number of descents) is a measure of the <u>presortedness</u> of the permutation.

The i^{th} element of an inversion vector of a permutation is the number of elements greater than i to the left of i

$$5\ \ 9\ \ 1\ \ 8\ \ 2\ \ 6\ \ 4\ \ 7\ \ 3$$

$$(2\ \ 3\ \ 6\ \ 4\ \ 0\ \ 2\ \ 2\ \ 1\ \ 0)$$

Summing up the inversion vector gives the number of inversions.

How many inversion vectors are there?

There can be from 0 to $N-1$ elements greater than 1
0 to $N-2$ elements greater than 2
$\vdots$
0 elements greater than N

Thus there can be $N!$ inversion vectors for the $N!$ permutations, and in fact there is a bijection between them!

$$\begin{matrix} 5 & 9 & 1 & 8 & 2 & 6 & 4 & 7 & 3 \\ (2 & 3 & 6 & 4 & 0 & 2 & 2 & 1 & 0) \end{matrix}$$

place the $N-i$th element to the left & skip the appropriate number of spots given by the inversion vector!

9
98
987
9867
59867
598647
5986473
59826473
591826473

$\boxed{\text{Thus both inversion vectors and cycle structures determines permutations!}}$

Bernoulli Numbers

As we have seen, the sum $S_m(n) = \sum_{k=0}^{n-1} k^m$ occurs frequently. These closed forms for the sum all have certain regularities:

$$
\begin{aligned}
S_0(n) &= n \\
S_1(n) &= \tfrac{1}{2}n^2 - \tfrac{1}{2}n \\
S_2(n) &= \tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{6}n \\
S_3(n) &= \tfrac{1}{4}n^4 - \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 \\
S_4(n) &= \tfrac{1}{5}n^5 - \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n \\
S_5(n) &= \tfrac{1}{6}n^6 - \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2 \\
S_6(n) &= \tfrac{1}{7}n^7 - \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n \\
S_7(n) &= \tfrac{1}{8}n^8 - \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2 \\
S_8(n) &= \tfrac{1}{9}n^9 - \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n \\
S_9(n) &= \tfrac{1}{10}n^{10} - \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2 \\
S_{10}(n) &= \tfrac{1}{11}n^{11} - \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - n^7 + n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n
\end{aligned}
$$

- coefficient of lowest order term is $1/m$

- alternate terms do not appear in the sum

These are in fact a function of the Bernoulli numbers:

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

We will <u>NOT</u> discuss this, but remember where you can look the sums up!