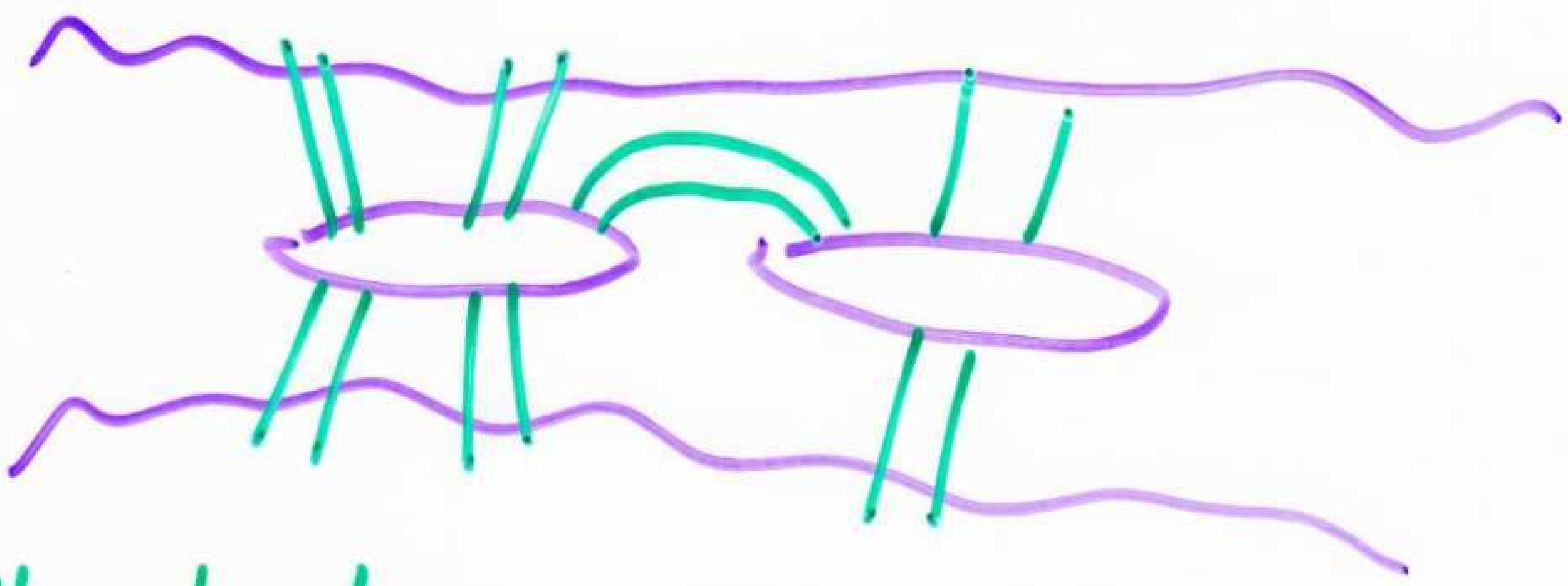
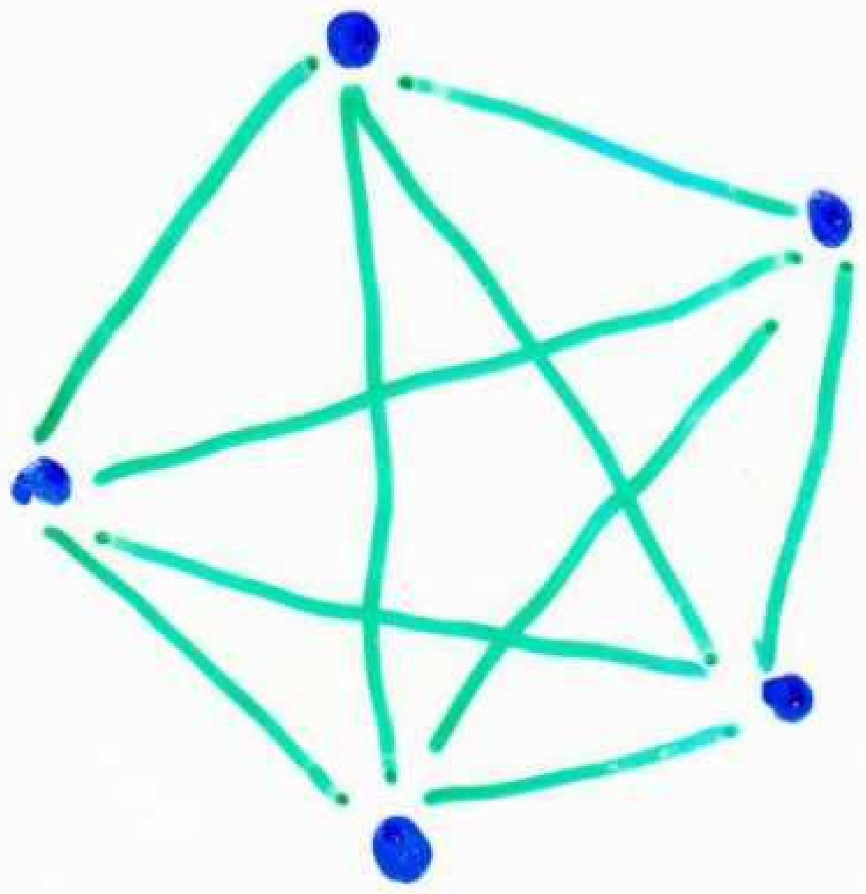
Eulerian Graphs

The city of Konigsberg used to have seven bridges:

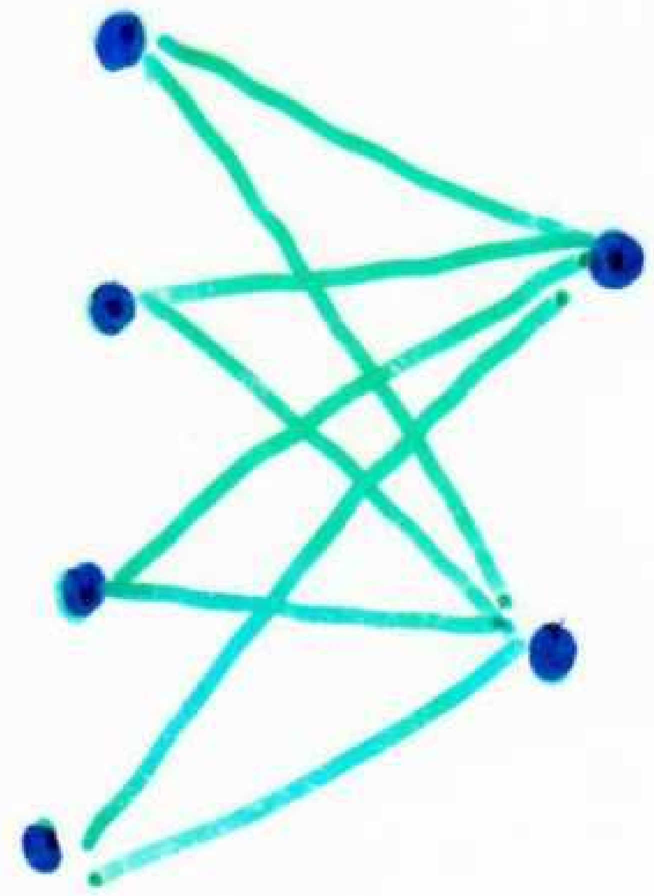


and the locals wanted to find a way to visit each bridge exactly once on a walk. In 1736, Euler founded graph theory by showing it can't be done.

An Eulerian cycle of a graph is a circuit which contains each edge exactly once. A graph is Eulerian if it contains an Eulerian cycle.



K_4



$K_{4,2}$

Eulerian Graphs

"Can be drawn without lifting the pen"

Theorem: A connected graph contains an Eulerian cycle iff each vertex is of even degree.

Proof: We will show the equivalence of three statements.

1. If a graph is Eulerian then each vertex is of even degree

If it Eulerian, it has an Eulerian cycle. This cycle contains each edge, and since the number of times a vertex is entered equals the number of times it is left, each vertex must be even.

2. If each vertex of a connected graph is even - the edges can partitioned into disjoint cycles.

Proof by induction on the number of cycles.

All even graphs with at most one cycle can be partitioned!

A graph without a cycle is a tree, so it has vertices of degree 1.

(continued)

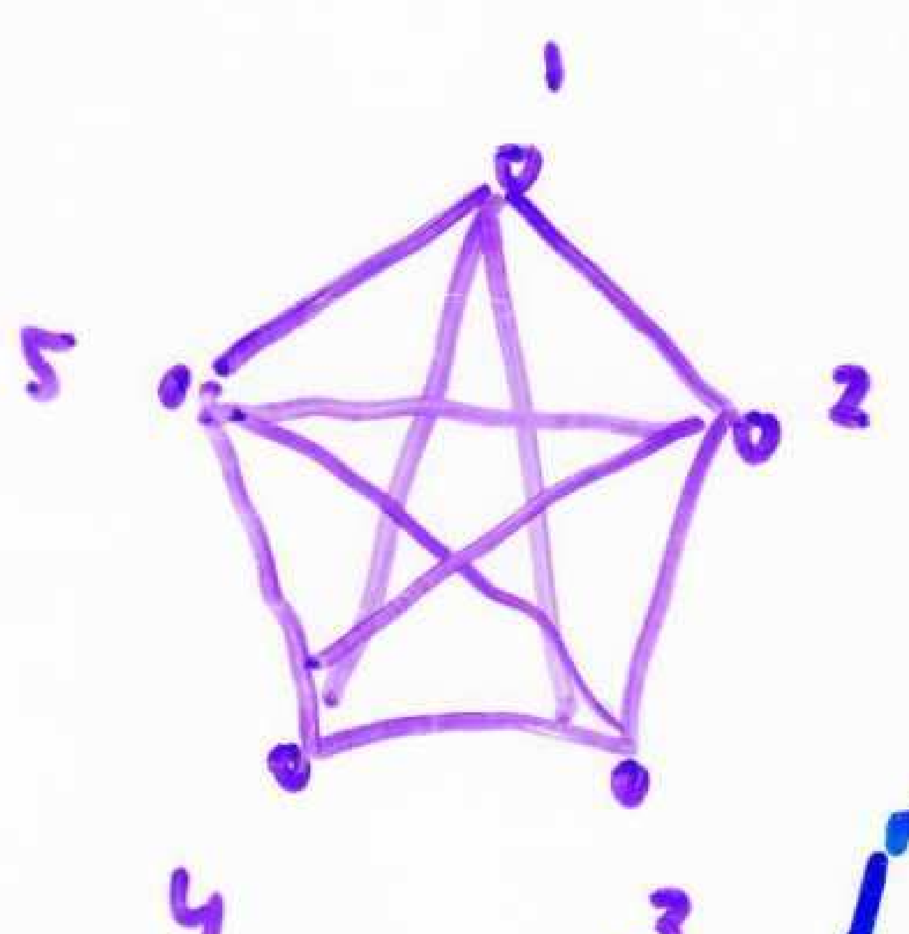
Assume that each even connected graph which can have up to $k-1$ cycles extracted can be partitioned into at most $k-1$ cycles.

For any graph of even degree with k cycles, deleting any cycle leaves a, possibly disconnected graph which must be of even degree. Each connected component contains at most $k-1$ cycles, so the result follows by induction.

3. A connected graph whose edges can be partitioned into disjoint cycles is Eulerian.

Since the graph is connected, the cycles can be ordered such that the k^{th} cycle shares at least one vertex with the union of the first $k-1$.

From the Tour of the first $k-1$ cycles, simply splice in the k^{th}:



$C_1 = 1\,2\,3\,1 \rightarrow E_1 = 1\,2\,3\,1$

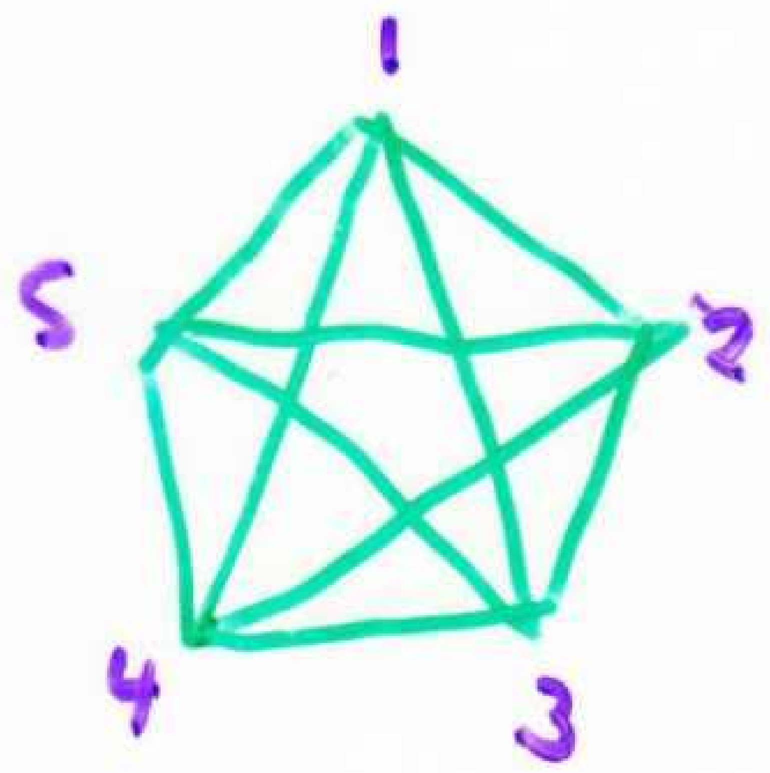
$C_2 = 2\,4\,5\,2 \rightarrow E_2 = 1\,2\,4\,5\,2\,3\,1$

$C_3 = 1\,4\,3\,5\,1 \rightarrow E_3 = 1\,4\,3\,5\,1\,2\,4\,5\,2\,3\,1$

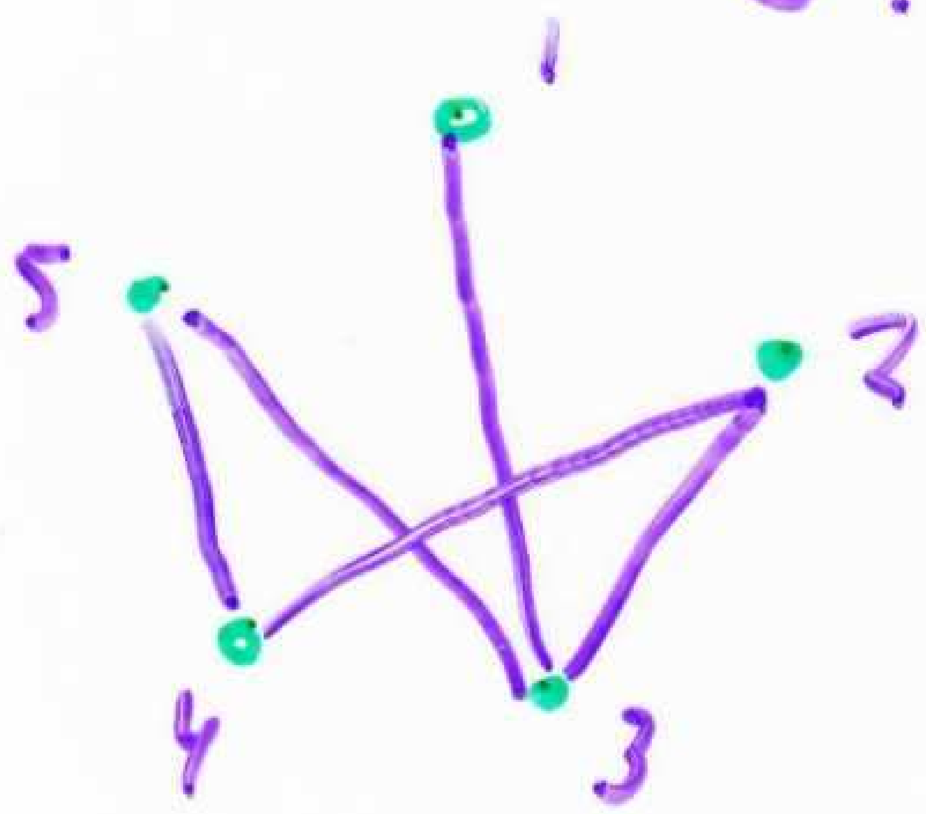
This gives an algorithm for finding Eulerian cycles!

Fleury's Algorithm

An alternate algorithm builds up the circuit sequentially – start at one vertex, pick an edge, walk there + delete the edge. At each step, avoid a <u>bridge</u> for as long as possible!



1 2 5 1 4 3 . . . .



So long as we avoid (1,3) we can finish the job.

The reason the algorithm works ... 2 4 5 3 1 is that Eulerian graphs have no bridges. When we cross a bridge there is no way to get back, so we must use it last.

At any point in the algorithm, there will be at most two vertices of odd degree, since that is an necessary + sufficient condition for an <u>Eulerian Path</u>.

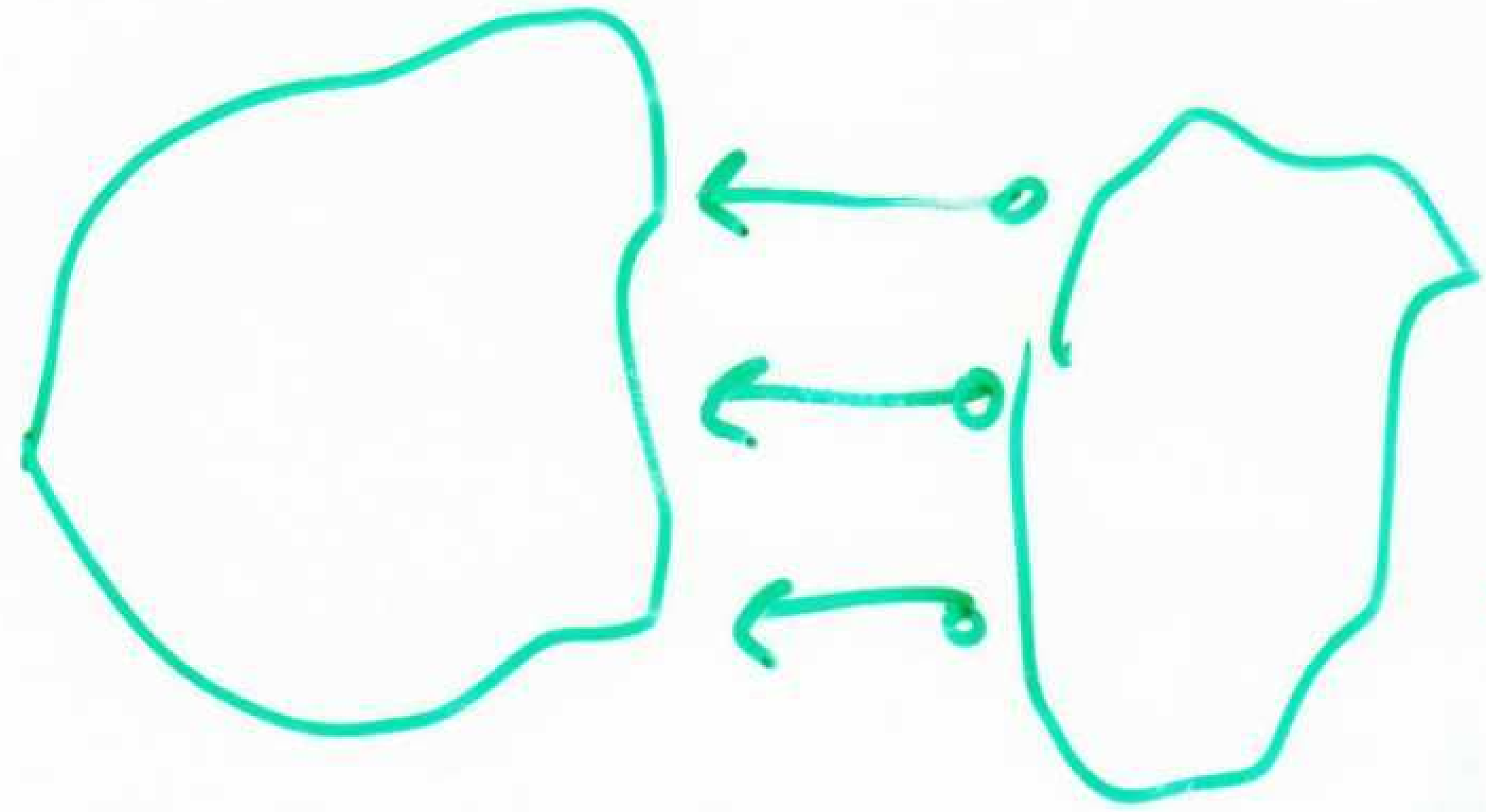
Directed Eulerian Graphs

For any weakly connected directed graph, the following statements are equivalent:

1. The Digraph has a directed Eulerian cycle
2. For each vertex, the indegree = outdegree
3. The edges can be partitioned into directed cycles.

Proof: (1) to (2) what goes in goes out
(2) to (3) by induction on the number of directed cycles
(3) to (1) same algorithm works. ∎

Clearly a graph must be strongly connected to be Eulerian - why does weakly connected suffice?



within each strongly connected component, the sum of the indegrees = outdegrees.

Debruijn Sequences

Suppose you want to crack a safe, whose combination consists of N symbols from a set $\{1 \ldots \sigma\}$. To try all combinations requires $N \times \sigma^N$ turns.

Now suppose the safe has no reset mechanism but opens when the last N symbols are the combination.

Ex: Combinations 011, 101 $\Rightarrow$ 01101
or
1011 ✓

What is the shortest string on Σ such that every word of N symbols appears somewhere in the string?

Such a sequence is a Debruijn sequence and can be found using Eulerian cycles!

Ex: $\{0,1\}$ $N=3$

$$\left\{\begin{matrix} 000, & 100 \\ 001, & 101 \\ 010, & 110 \\ 011, & 111 \end{matrix}\right\}$$

Debruijn sequence!

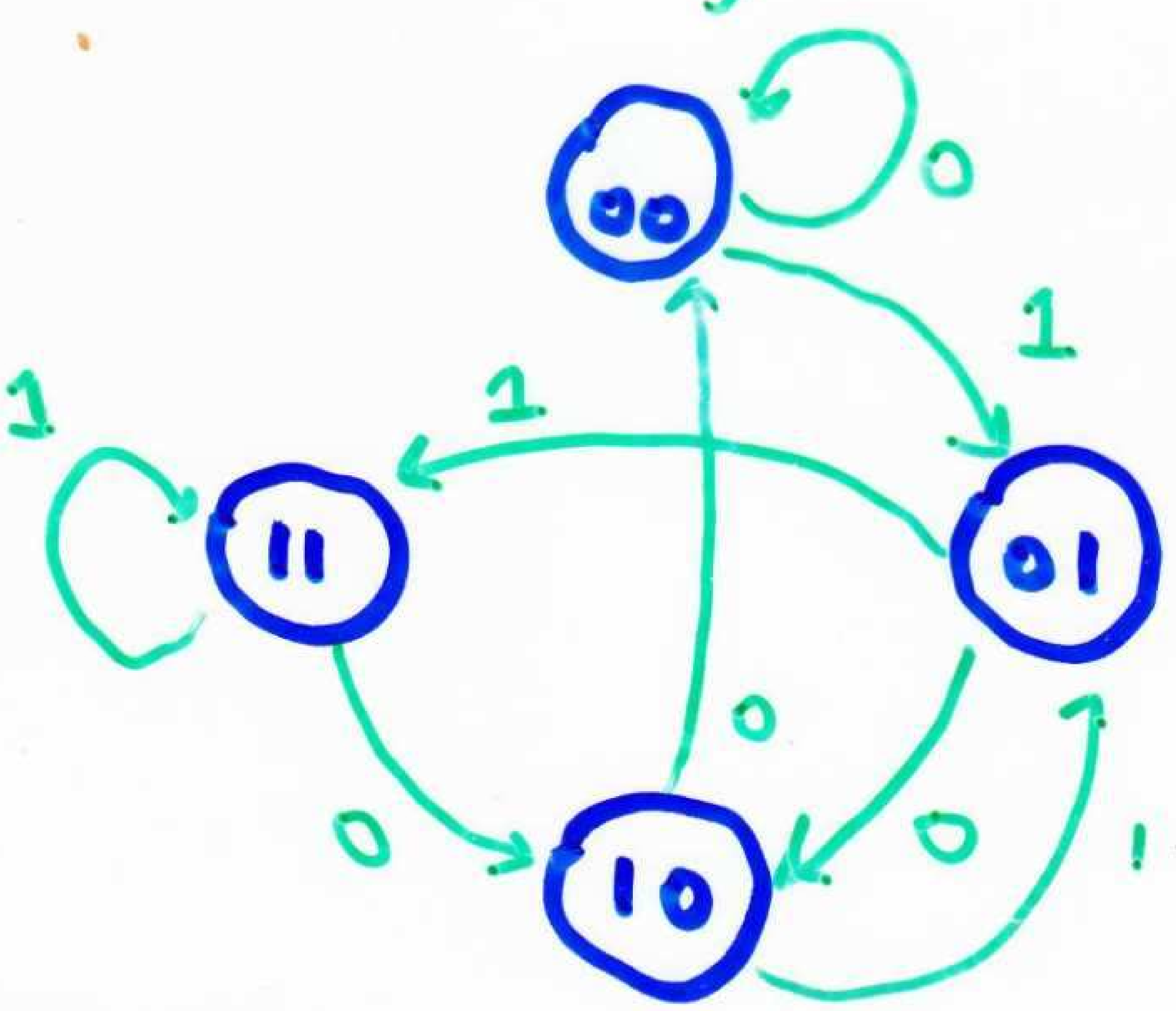
00011101

Note: this is a circular sequence!

We can find Debruijn sequences by building a special graph.

The vertices will be strings of length $n-1$

The edges will correspond to what the last $n-1$ symbols of the string is if we append each symbol to the vertex string:



Each vertex has out-degree σ, by symmetry must have equal in degrees, so the graph is Eulerian

Each edge is a string of length n - take a tour and output the transition symbol:

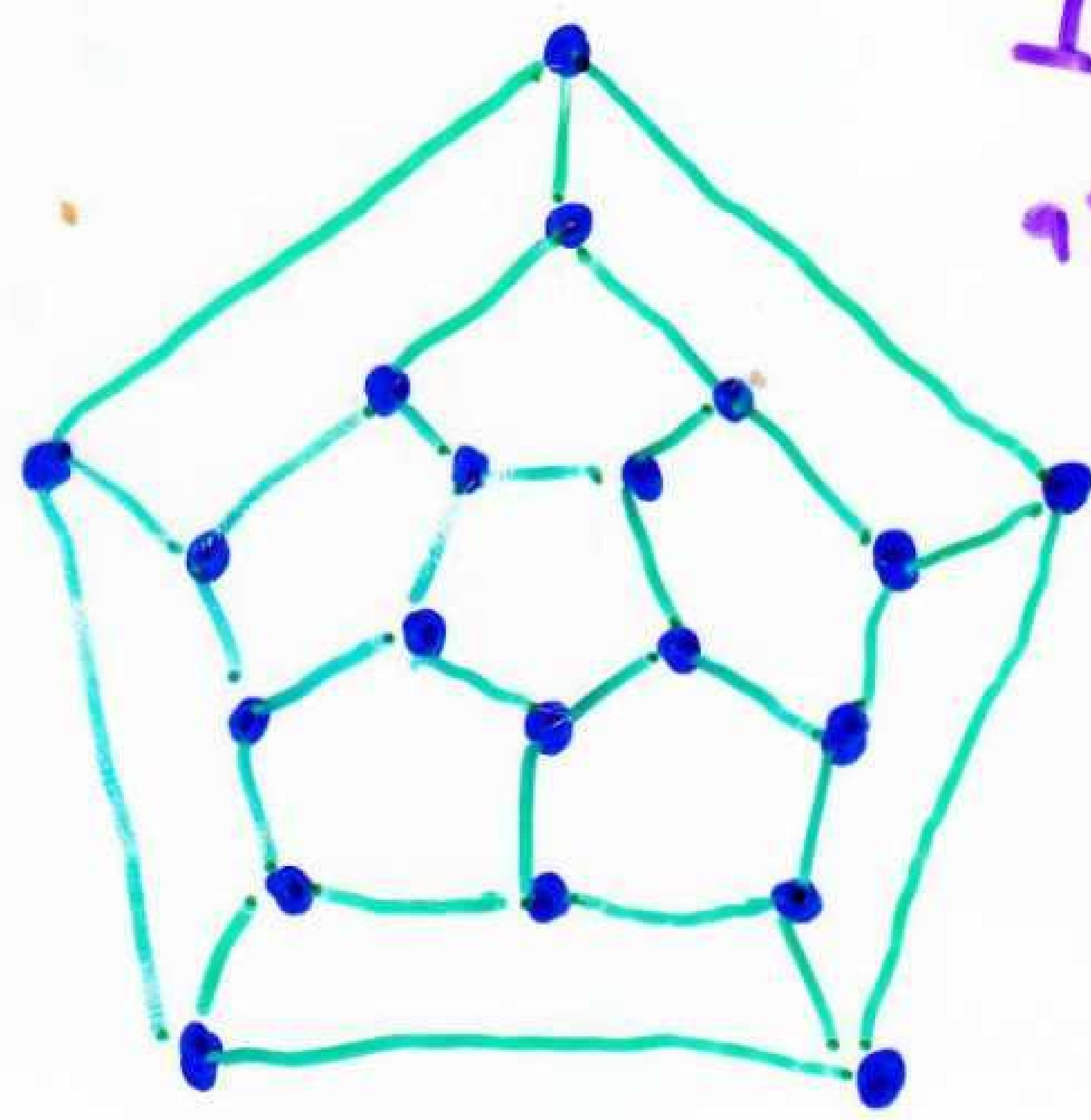
$00 \to 00 \to 01 \to 10 \to 01 \to 11 \to 10 \to 00 \to 00$

0 1 0 1 1 1 → 0 → 0

Hamiltonian Cycles

An Eulerian cycle visits each edge without repetition.

A Hamiltonian cycle visits each vertex exactly once without repetition.



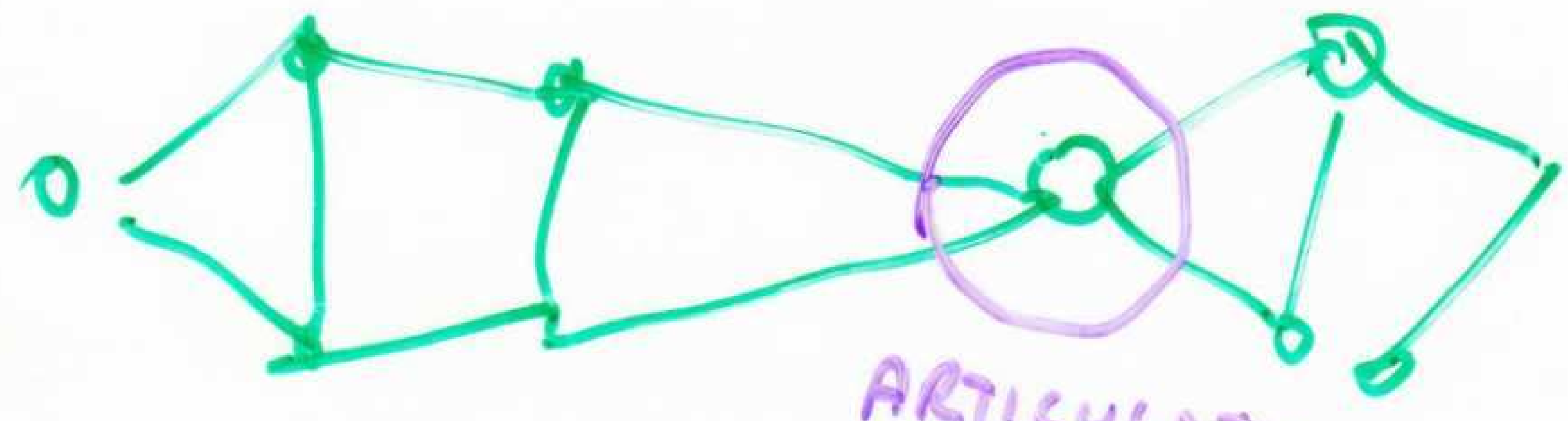
In 1865, Sir William Hamilton attempted to market a game based on finding a Hamiltonian tour of a dodecahedron. Can you find it?

Euler's knights tour problem dates to 98 years before Hamilton's game, but "Eulerian Cycle" was already taken!

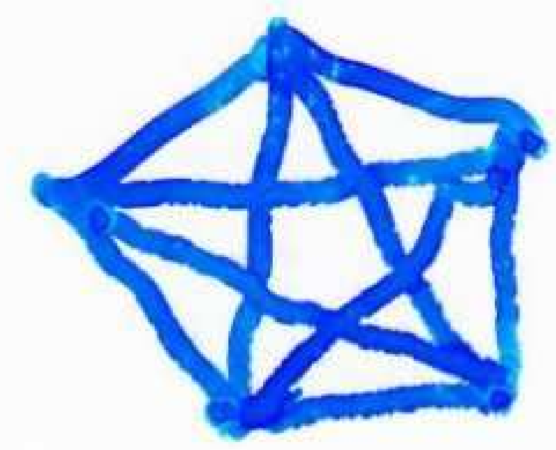
Unlike for Eulerian cycles, no necessary **AND** sufficient condition for a graph to be Hamiltonian is known. Likewise no polynomial time algorithm for testing this is known and the problem is NP-complete.

Hamiltonian cycle is a special case of the traveling Salesman problem.

All Hamiltonian graphs are biconnected, so there is a necessary condition:



Any Complete graph is Hamiltonian, since any edge we need is present.

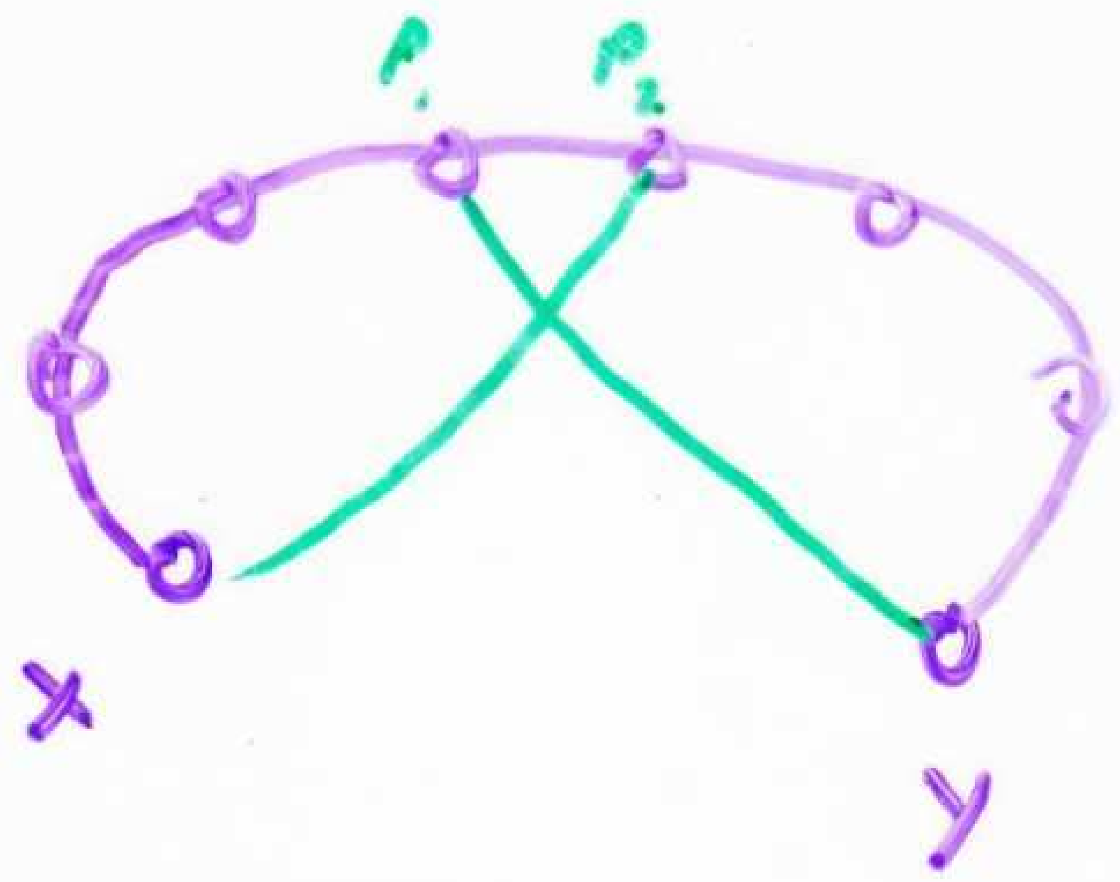


This suggests that if we have enough edges, the graph is certain to be Hamiltonian, but how much is enough?

Theorem: If G is a graph on at least $n \geq 3$ vertices such that for all non adjacent vertices x, y, degree x + degree $y \geq n$, G is Hamiltonian.

Proof: Assume the converse. Then there exists a graph which is not Hamiltonian but each pair of non-adjacent vertices has degrees totally at least n.

By adding edges to this graph we can get a graph so adding any extra edge makes it Hamiltonian. Thus adding edge (x, y) makes it Hamiltonian & means there exists a Hamiltonian Path from x to y



Observe there cannot be vertices p_1, p_2 s.t. $(p_1, p_2), (x, p_2), (y, p_1)$ are all edges, or else it is Hamiltonian.

Thus, there are at least degree x vertices y cannot be adjacent to, so degree $y \leq n - 1 -$ degree x which is a contradiction!

"x" "x inadjacent to"

As a corollary, observe that if the smallest degree vertex is $\geq n/2$, the graph is Hamiltonian.