# Predicate Logic
# PART 2

CSE 352 Artificial Intelligence

Professor Anita Wasilewska

Lecture Notes

# Predicate Logic Part 2

- PREDICATE LOGIC TAUTOLOGIES
- BASIC LAWS OF QUANTIFIERS

# Basic Laws of Quantifiers (Predicate Logic Tautologies)

**De Morgan Law**

**$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$**

**$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$**

where A(x) is any formula with free variable x

≡ means “logically equivalently”

**Definability:**

**$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$**

**$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$**

**Application Example**: A(x) is $((Px) \wedge \neg R(x)) \neg Q(x,y))$

$\equiv \exists x(P(x) \wedge \neg R(X) \wedge \neg Q(x,y))$

# Example (Mathematical Formula)

Laws Application:

**$\neg \forall x((x>0 \Rightarrow x+y > 0) \land \exists y\ (y>0))$**

$\equiv$ (by De Morgan's Law)

$\exists x((x>0 \land x+y > 0) \land \exists y\ (y > 0))$

**$\equiv \exists x((x>0 \land x+y \leq 0) \lor \forall y(y \geq 0))$**

$\neg\ (A \Rightarrow B) \equiv (A \land \neg B),\ \neg(A \land B) \equiv (\neg A \lor \neg B)$

$\neg(x + y) > 0) \equiv x+y \leq 0$

$\neg\ \exists y\ (y >< 0) \equiv \forall y\ \neg\ (y < 0)$

$\equiv \exists y\ (y \geq 0)$

**Logic Formula** (corresponding to the Math formula

**$\neg\ \forall x(A(x) \Rightarrow B(\ x,y)) \land \exists y\ C(y))$**

$\equiv \exists x\ \neg((A(x) \Rightarrow B(x,y) \land \exists y\ C(y))$

$\equiv \exists x((A(x) \land \neg B(\ \ x,y)\ ) \lor \neg\ \exists y\ C(y))$

**$\equiv \exists x\ ((A(x) \land \neg B(\ x,y)) \lor \forall y\ \neg C(y))$**

# Distributivity Laws (to be proved)

1. **$\exists x(A(x) \vee B(x)) \equiv (\exists x\, A(x) \vee B(x))$**
   Existential quantifier is distributive over $\vee$ ($\exists x$, $\vee$)
2. **$\forall x\, (A(x) \wedge B(x)) \equiv (\forall x\, A(x) \wedge B(x))$** universal quantifier is distributive over $\wedge$ ($\forall x$ , $\wedge$)
3. Existential quantifier is distributive over $\wedge$ in only one direction

**$\exists x(A(x) \wedge B(x)\,) \Rightarrow (\exists x\, A(x) \wedge \exists x\, B(x))$**

It is not true, that for any $x \neq \phi$ and any A(x), B(x) $(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x)\,)$

**Example:** $x \in R$ for x = R $A(x) > 0$, $B(x) = x^2$

$\exists x\, (x>0) \wedge \exists x(x>0)$ is a **true** statement!

$\exists x(x>0 \wedge x< 0)$ is **false**!

# Distributivity (continued)

4. Universal quantifier is distributive over $\wedge$ in only one direction:

**$((\forall x\ A(x) \vee \forall x\ B(x)) \Rightarrow \forall x(A(x) \vee B(x)))$**

**$A(x) = x < 0 \quad B(x) = x \geq 0$**

**Interpretation $T \Rightarrow F = F$**

**Example**: $x \in R$ for $x = R$

$\forall x\ (x>0 \vee x \geq 0)$ is **true**

$\forall x(x<0) \vee \forall x(x \geq 0)$ **false**

5. Universal quantifier is distributive over $\Rightarrow$ in one direction:

**$\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\ A(x) \Rightarrow \forall x\ B(x))$**

Example: Take $x \in R$

$(\forall x(x < 0\ ) \Rightarrow \forall x(x+1 > 0)$ is False

Take $x= -2$, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

# Introduction and Elimination Laws

B- Formula without free variables

**6.** $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$

**7.** $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$

**8.** $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$

**9.** $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

**10.** $\forall x(A(x) \vee B) \equiv (\forall x\, A(x) \vee B)$

**11.** $\forall x(A(x) \wedge B) \equiv (\forall x\, A(x) \wedge B)$

**12.** $\exists x(A(x) \vee B) \equiv (\exists x\, A(x) \vee B)$

**13.** $\exists x(A(x) \wedge B) \equiv (\exists x\, A(x) \wedge B)$

Remark: we prove 6 -9 from 10 – 13 + de Morgan + definability of implication

# Intuitive (not very formal) Semantics for Predicate Logic

We can use truth sets for predicates $x \neq \phi$

**$\{x \in X: P(x)\}$** is called a truth set for the predicate $P(x)$.

**Example1**:

$P(x): x+1 = 3$ **interpretation of $P(x)$** in x $x=\{3, 4\}$

$x=\{1, 2, 3\}$

$\{x \in X: P(x)\} = 2$, $\{x \in X: P(x)\} = \phi$

**Example2**:

$P(x): x^2 \leq 0$ **Interpretation of $P(x)$**

$x = N$ $x = P^{+}-\{0\}$

$\{x: P(x) = \{0\}$ $\{x:P(x)\} = \phi$

# Intuitive ( not very formal) semantics for Predicate Logic

We use truth sets for predicates $x \neq \phi$

**Conjunction:**

$\{x \in X: (P(x) \wedge Q(x))\} = \{x: P(x) \wedge \{x: Q(x)\}$

$\wedge$ is used to symbolize conjunction, known as an intersection

**Disjunction:**

$\{x \in X: (P(x) \vee Q(x))\} = \{x: P(x) \vee \{x: Q(x)\}$

$\vee$ is used for disjunction, known as union

**Negation:**

$\{x \in X: \neg P(x)\} = X - \{x \in X: P(x)\}$

$\neg$ is the negation and – is the complement

# Intuitive ( not very formal) semantics for Predicate Logic

**Implication:**

$\{x \in X: (P(x) \Rightarrow Q(x))\} \equiv X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= - x:P(x)\} \vee \{x : Q(x)\}$

$=\{x: \neg P(x)\} \vee \{x : Q(x)\}$ Interpretation

**Example:**

$\{x \in N\ n>0 \Rightarrow n^2 <0\} = \{x \in N\ x \leq 0\} \vee \{\{x \in N : n^2<0\}$

$=\{0\} \vee \phi = \{0\}$

# Truth Sets for Quantifiers

**Definition:**

**$\forall x\, A(x) = T$ iff $\{x \in X: A(x)\} = X$ in the domain $X \neq \phi$ where $A(x)$ is any formula with x-free**

**Definition:**

**$\forall x\, A(x) = F \quad (X \neq \phi)$**

**iff $\{x \in X: A(x)\} \neq X$**

**where $A(x)$ is any formula with x-free variable**

# Truth Sets for Quantifiers

**Definition:**

$\exists x\, A(x) = T$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} \neq \phi$

**Definition:**

$\exists x\, A(x) = F$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} = \phi$

**A(x) is a formula (complex) with free variable x.**

Remark

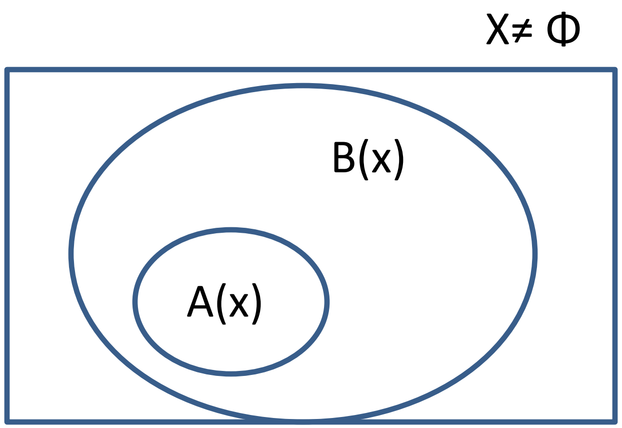
Observe that

$\forall x\,(A(x) \Rightarrow B(x)) = T \quad X \neq \Phi$

Iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff $\{x : A(x)\} \subseteq \{x : B(x)\}$

Picture



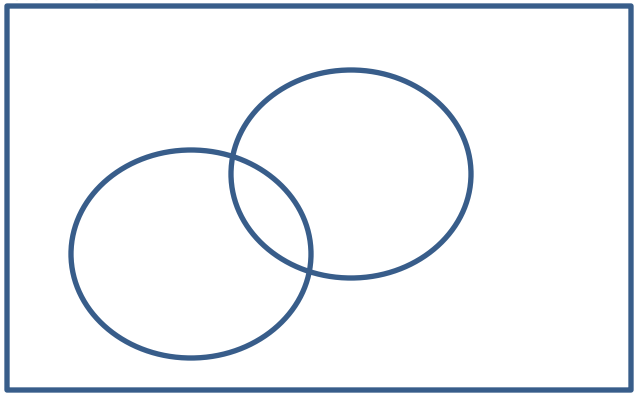

Venn Diagrams For Quantifiers

# Venn Diagrams For Quantifiers

**$\exists x(A(x) \wedge B(x))=T$** iff $X \neq \Phi$

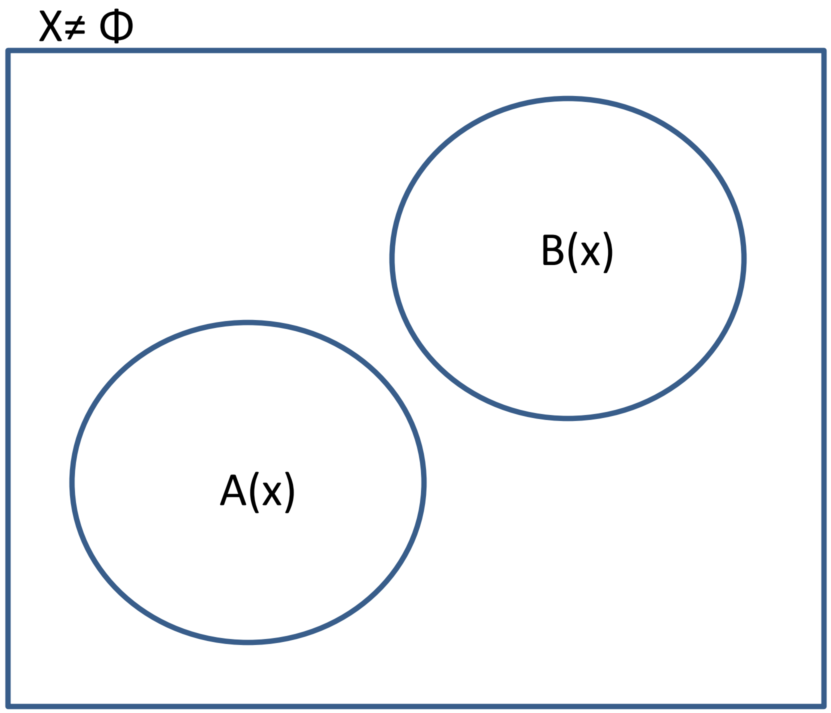
$\{x:A(X)\} \wedge \{x:B(x)\} \neq \Phi$

**Picture**



$\exists x(A(x) \wedge B(x)) = F$ iff $\{x:A(x) \wedge \{x:B(x)\} = \Phi$

# Pictures




Remember {x:A(x)}, {x:b(x)}
Can be Φ!

# Example:

Draw a picture for a situation where (in $X \neq \Phi$)

1. **$\exists x\, P(x) = T$,**
2. **$\exists x\, Q(x) = T$,**
3. **$\exists x (P(x) \wedge Q(x)) = F$ and**
4. **$\forall x\, (P(x) \vee Q(x) = F$**

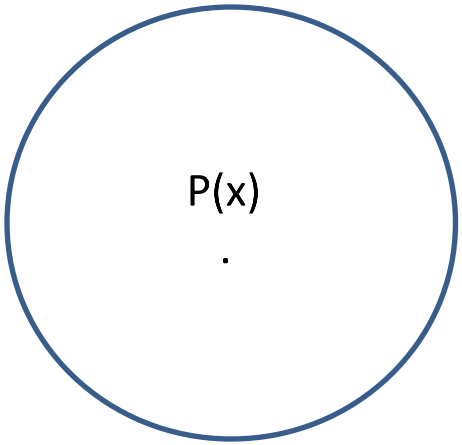
1. $\exists x\, P(x) = T$ iff $\{ x{:}P(x)\} \neq \Phi$

2. $\exists x\, Q(x) = T$ iff $\{x{:}Q\,(x)\} \neq \Phi$

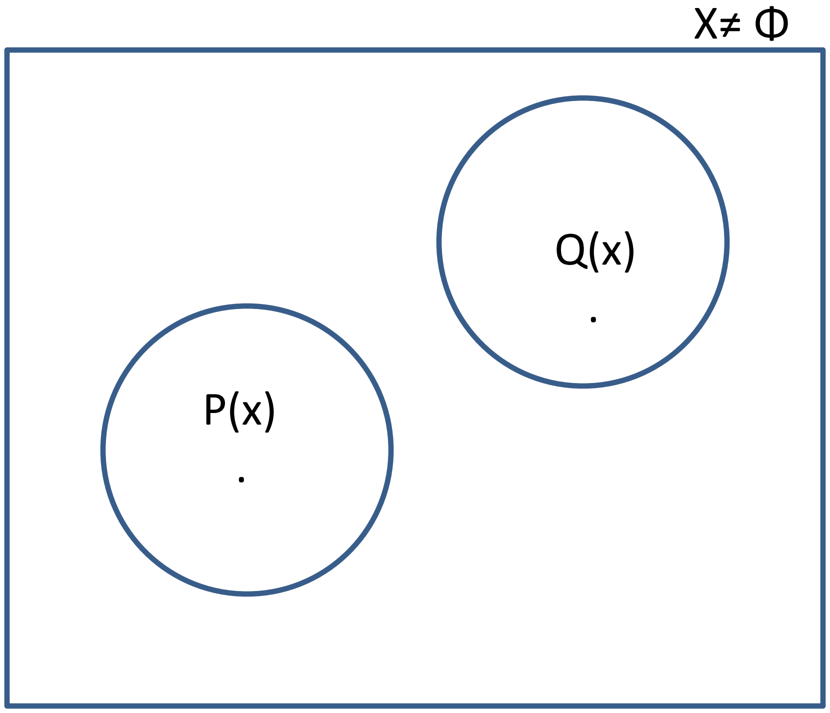
3. $\{x{:}P(x)\} \wedge \{x{:}Q(x)\} \neq \Phi$

4. $\{x{:}P(x)\} \vee \{ x{:}Q(x)\} \neq X$

# Picture:




Denotes P(x) ≠ Φ



# Proving Predicate Tautologies

Prove that

$\vdash(\forall x\ A(x) \Rightarrow \exists x\ A(x))$

Proof:

Assume that not True

(Proof by contradiction) i.e. that there are $X \neq \Phi, A(x)$ such that.

$(\forall x\ A(x) \Rightarrow \exists x\ A(x))$ **= is F**

iff **$\forall x\ A(x)$=T** and **$\exists x\ A(x)$=F** $(A \Rightarrow B) = F$

iff (def) $x \neq \phi$ iff (def)

**$\{x \in X : A(x)\} = X$** and **$\{x \in X : A(x)\} = \phi$**

**iff $x = \phi$**

Both of these formulas are a contradiction with **$x \neq \phi$** Hence proved!!

Prove:

**$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$**

$\exists x\, \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ when $B = \phi$ then $B \neq x$ as $x \neq \phi$ $B \neq \phi$

$X - \{x: A(x)\} \neq \phi$

**iff** $X - B \neq \phi$

$\{x: A(x)\} \neq x$ **iff**

**iff** $B \neq x$

$\forall x\, A(x) = F$ we assume that for any A(x)

**iff** $\{x \in X: A(x)\}$ exists

$\neg \forall x\, A(x) = T$

Example: B – No variable x

**$\forall x\ (A(x) \vee B) \equiv \forall x\ A(X) \vee B$**

= T **iff**

{x: A(x) ∨ {x: B} = X

**iff** {x: A(x) = X

or B=X

it means

$\forall x\ \ A(x)=T$ or $B=T$

and

$\forall x\ (A(x) \vee B)\ = T$

Prove

**$\exists x(A(x) \wedge B(x)) \equiv \exists x\, A(x) \wedge \exists x\, B(x)$**

$\exists x(A(x) \wedge B(x)\} = T$ iff

$\{x: (A(x) \wedge B(x)\} \neq \phi$ (definition)

$= \{x: (A(x)\} \wedge \{x: (B(x)\} \neq \phi$ iff $A \wedge B \neq \phi$

iff $A \neq \phi$ and $B \neq \phi$

$\{x: A(x)\} \neq \phi$ and $\{x: B(x)\} \neq \phi$

$= \exists x\, A(x)=T$ and $\exists x\, B(x)=T$