SOME PROBLEMS: chapters 5,6

Reminder: We define **H** semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = max\{a, b\}, \qquad a \cap b = min\{a, b\}.$$

The Truth Tables for Implication and Negation are:

H-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

QUESTION 1 **We know that**

$$v : VAR \longrightarrow \{F, \bot, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \bot$$

under **H** semantics. **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

Solution : $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics if and only if (we use shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ if and only if $a = T, b = T$ and $(T \Rightarrow c) = \perp$ if and only i f $c = \perp$. I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp \quad iff \quad a = T,\ b = T, c = \perp .$$

Now we can we **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows (in shorthand notation).
$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) =$
$(((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T)) =$
$((T \Rightarrow (T \Rightarrow F)) \cup T) = T$.

We define a 4 valued Ł_4 logic semantics as follows. The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$.

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of Ł_4 as the following operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$.

Negation $\neg : \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that

$$\neg\bot_1 = \bot_1, \ \neg\bot_2 = \bot_2, \quad \neg F = T, \quad \neg T = F.$$

Conjunction $\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \ \bot_1, \bot_2, , T\}$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cap b = min\{a, b\}.$$

Disjunction $\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cup b = max\{a, b\}.$$

Implication $\Rightarrow: \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

QUESTION 2

Part 1 Write all TTables for **Ł**$_4$

Solution :

Ł$_4$ Negation

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	$\bot_1$	$\bot_2$	F

Ł$_4$ Conjunction

$\cap$	F	$\bot_1$	$\bot_2$	T
F	F	F	F	F
$\bot_1$	F	$\bot_1$	$\bot_1$	$\bot_1$
$\bot_2$	F	$\bot_1$	$\bot_2$	$\bot_2$
T	F	$\bot_1$	$\bot_2$	T

Ł$_4$ Disjunction

$\cup$	F	$\perp_1$	$\perp_2$	T
F	F	$\perp_1$	$\perp_2$	T
$\perp_1$	$\perp_1$	$\perp_1$	$\perp_2$	T
$\perp_2$	$\perp_2$	$\perp_2$	$\perp_2$	T
T	T	T	T	T

Ł$_4$-Implication

$\Rightarrow$	F	$\perp_1$	$\perp_2$	T
F	T	T	T	T
$\perp_1$	$\perp_1$	T	T	T
$\perp_2$	$\perp_2$	$\perp_2$	T	T
T	F	$\perp_1$	$\perp_2$	T

Part 2 Verify whether

$$\models_{Ł_4}((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution : Let v be a truth assignment such that $v(a) = v(b) = \bot_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg \bot_1 \cup \bot_1)) = (T \Rightarrow (\bot_1 \cup \bot_1)) = (T \Rightarrow \bot_1) = \bot_1$.

This proves that v is a counter-model for our formula and

$$\not\models_{Ł_4}((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a v such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg\bot_2 \cup \bot_2)) = (T \Rightarrow (\bot_2 \cup \bot_2)) = (T \Rightarrow \bot_2) = \bot_2$.

QUESTION 3 Prove using proper logical equivalences (list them at each step) that

1. $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$,

Solution: $\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$ $\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))$.

2. $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$.

Solution: $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B)) \equiv^{deMorgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$
$\equiv^{dneg} ((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B))$.

QUESTION 4 We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:
$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.
We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that

$$A \equiv B,$$

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that

$$C \equiv D.$$

Prove that $\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$.

Solution: The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$