

# **Chapter 8: Hilbert Proof Systems and Deduction Theorem**

## **PROOF OF THE DEDUCTION THEOREM**

## Hilbert System $H_1$ :

$$H_1 = ( \mathcal{L}_{\{\Rightarrow\}}, \mathcal{F} \{A1, A2\} \text{ MP} )$$

$$\mathbf{A1} \quad (A \Rightarrow (B \Rightarrow A)),$$

$$\mathbf{A2} \quad ((A \Rightarrow (B \Rightarrow C)) \\ \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

**MP**

$$(MP) \frac{A ; (A \Rightarrow B)}{B},$$

## DEDUCTION THEOREM (Herbrand, 1930)

For any formulas  $A, B$ ,

*if*  $A \vdash B$ , *then*  $\vdash (A \Rightarrow B)$ .

We are going to prove now that for our system  $H_1$  is strong enough to prove the Deduction Theorem for it. In fact we prove a more general version of Herbrand theorem. To formulate it we introduce the following notation.

**We write**

$$\Gamma, A \vdash B$$

for

$$\Gamma \cup \{A\} \vdash B$$

**In general** we write

$$\Gamma, A_1, A_2, \dots, A_n \vdash B$$

for

$$\Gamma \cup \{A_1, A_2, \dots, A_n\} \vdash B.$$

**Deduction Theorem for  $H_1$**  For any subset  $\Gamma$  of the set of formulas  $\mathcal{F}$  of  $H_1$  and for any formulas  $A, B \in \mathcal{F}$ ,

$$\Gamma, A \vdash_{H_1} B \text{ if and only if } \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \text{ if and only if } \vdash_{H_1} (A \Rightarrow B).$$

# PROOF OF DEDUCTION THEOREM

We use in the proof the symbol  $\vdash$  instead of  $\vdash_{H_1}$ .

**Part 1.** We first prove:

*If  $\Gamma, A \vdash B$  then  $\Gamma \vdash (A \Rightarrow B)$ .*

**Assume that**

$$\Gamma, A \vdash B,$$

i.e. that we have a formal proof

$$B_1, B_2, \dots, B_n$$

of  $B$  from the set of formulas  $\Gamma \cup \{A\}$ , we have to show that

$$\Gamma \vdash (A \Rightarrow B).$$

**In order** to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from  $\Gamma, A \vdash B$ , we prove that a stronger statement, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for any  $B_i$  ( $1 \leq i \leq n$ ) in the formal proof  $B_1, B_2, \dots, B_n$  of  $B$  also follows from  $\Gamma, A \vdash B$ .

**Hence** in particular case, when  $i = n$ , we will obtain that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from  $\Gamma, A \vdash B$ , and that will end the proof of Part 1.

**The proof** of Part 1 is conducted by induction on  $i$  for  $1 \leq i \leq n$ .

**Step**  $i = 1$  (base step).

**Observe** that when  $i = 1$ , it means that the formal proof

$$B_1, B_2, \dots, B_n$$

contains only one element  $B_1$ .

**By the definition** of the formal proof from  $\Gamma \cup \{A\}$ , we have that

**(1)**  $B_1$  is a logical axiom, or  $B_1 \in \Gamma$ , or

**(2)**  $B_1 = A$ .

**This means** that  $B_1 \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$ .

**Now we have** two cases to consider.

**Case 1:**  $B_1 \in \{A1, A2\} \cup \Gamma$ .

**Observe** that

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is the axiom  $A1$  and by assumption

$$B_1 \in \{A1, A2\} \cup \Gamma.$$

**We get** the required proof of  $(A \Rightarrow B_1)$  from  $\Gamma$  by the following application of the Modus Ponens rule

$$(MP) \frac{B_1 ; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

**Case 2:**  $B_1 = A$

**When**  $B_1 = A$  then to prove  $\Gamma \vdash (A \Rightarrow B)$   
means to prove

$$\Gamma \vdash (A \Rightarrow A),$$

what holds by the monotonicity of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A).$$

**The above** cases conclude the proof of

$$\Gamma \vdash (A \Rightarrow B_i)$$

for  $i = 1$ .

## INDUCTIVE STEP

**Assume** that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all  $k < i$ ,

**we will show** that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i).$$

**Consider** a formula  $B_i$  in the formal proof  
 $B_1, B_2, \dots, B_n$

**By the definition** of the formal proof we have the following

**Case 1**  $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$  or

**Case 2:**  $B_i$  follows by MP from certain  $B_j, B_m$  such that  $j < m < i$ .

**We have** to consider these cases.

**Case 1:**

$$B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}.$$

**The proof** of  $(A \Rightarrow B_i)$  from  $\Gamma$  in this case is obtained from the proof of the **Step**  $i = 1$  by replacement  $B_1$  by  $B_i$  and will be omitted here as a straightforward repetition.

## Case 2:

$B_i$  is a conclusion of MP.

If  $B_i$  is a conclusion of MP, then we must have two formulas  $B_j, B_m$  in the formal proof

$B_1, B_2, \dots, B_n$  such that  $j < m < i$  and

$$(MP) \frac{B_j ; B_m}{B_i}.$$

**By the inductive** assumption,

the formulas  $B_j, B_m$  are such that

$$\Gamma \vdash (A \Rightarrow B_j)$$

and

$$\Gamma \vdash (A \Rightarrow B_m).$$

**Moreover**, by the definition of Modus Ponens rule, the formula  $B_m$  has to have a form

$$(B_j \Rightarrow B_i),$$

i.e.

$$B_m = (B_j \Rightarrow B_i),$$

and the inductive assumption can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

, for  $j < i$ .

**Observe now** that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A2 and hence **has a proof** in our system.

**By the monotonicity** of the consequence, it also has a proof from the set  $\Gamma$ , i.e.

**We know** that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))).$$

**Applying** the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i)) ; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)).$$

**Applying again** the rule MP i.e. performing the following

$$\frac{(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the inductive step.

**By the mathematical induction** principle, we hence have proved that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for all  $i$  such that  $1 \leq i \leq n$ .

**In particular** it is true for  $i = n$ , what means for  $B_n = B$  .

**This ends the proof** of the fact that

if  $\Gamma, A \vdash B$ , then  $\Gamma \vdash (A \Rightarrow B)$ .

**The proof** of the inverse implication:

*If*  $\Gamma \vdash_{H_1} (A \Rightarrow B)$ , *then*  $\Gamma, A \vdash_{H_1} B$

is straightforward and goes as follows.

**Assume that**

$\Gamma \vdash (A \Rightarrow B)$ .

**By the monotonicity** of the consequence we have also that

$\Gamma, A \vdash (A \Rightarrow B)$ .

**Obviously**

$\Gamma, A \vdash A$ .

**Applying** Modus Ponens to the above, we get the proof of  $B$  from  $\{\Gamma, A\}$  i.e.

**we have proved** that

$$\Gamma, A \vdash B.$$

**THIS ENDS** the proof of the deduction theorem for any set  $\Gamma \subseteq \mathcal{F}$  and any formulas  $A, B \in \mathcal{F}$ .

**The particular case** of the theorem is obtained from the above by assuming that the set  $\Gamma$  is empty.