Modes of Truth

In classical (propositional or predicate) logic, formulas are either true or false. In many applications, though, it makes sense to distinguish between various “modes” of truth, such as *necessarily true*, *known to be true*, or *believed to be true*.

Modal logics are important, for instance, in artificial intelligence for modelling scenarios with several interacting agents.

In reasoning about computation, it is often necessary to distinguish between truth at different points in time or to explore different possible futures. This leads to temporal logics like CTL.

We next introduce another logical system, called *basic modal logic*, by extending propositional logic by two new unary connectives, $\Box$ and $\Diamond$ (read “box” and “diamond,” respectively).

Syntax of Basic Modal Logic

The syntax of modal logic is defined by the following rules:

$$\begin{aligned} \phi \quad ::= \quad & \top \mid \bot \mid P \mid (\neg\phi) \\ & \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\ & \mid \Box\, \phi \mid \Diamond\, \phi \end{aligned}$$

Depending on the application, common readings of $\Box$ are "it is necessary that" or "it will always be true that," while $\Diamond$ may be intuitively interpreted as "it is possible that" or "it will eventually be true that."

In some applications just one of the two operators is used. Sometimes different variations of the operators are used.

Kripke Models

The semantics of modal logic is defined in terms of a collection W of "possible worlds" that are related to each other by an "accessibility" relation, which is simply a binary relation on W. Typically, certain properties (reflexivity, transitivity, etc.) are assumed of this relation, often depending on the application domain.

Definition

> Let A be a set of atomic formulas of a propositional logic. A *(Kripke) model* $\mathcal{M}$ is a triple (W, R, L), where
>
> 1. W is a set, the elements of which are called *(possible) worlds*,
> 2. R is a binary relation on W (i.e., $R \subseteq W \times W$) called the *accessibility* or *successor* relation, and
> 3. L is a function from W to $\mathcal{P}(A)$, called the *labelling function*.

We say that w' is *related* to w, or a *successor world* of w if wRw' (i.e., $(w, w') \in R$).

Note that CTL models, as defined previously, are Kripke models in this sense.

Semantics of Modal Operators

Let $\mathcal{M} = (W, R, L)$ be a Kripke model. We define a corresponding satisfaction relation $x \models \phi$, for worlds $x \in W$ and formulas ϕ by structural induction:

- $x \models \top$ and $x \not\models \bot$
- $x \models p$ iff $p \in L(x)$.
- $x \models \neg\phi$ iff $x \not\models \phi$.
- $x \models \phi_1 \wedge \phi_2$ iff $x \models \phi_1$ and $x \models \phi_2$.
- $x \models \phi_1 \vee \phi_2$ iff $x \models \phi_1$ or $x \models \phi_2$.
- $x \models \phi_1 \rightarrow \phi_2$ iff $x \not\models \phi_1$ or $x \models \phi_2$.
- $x \models \Box\phi$ iff $y \models \phi$, for all $y \in W$ with xRy.
- $x \models \Diamond\phi$ iff $y \models \phi$, for some $y \in W$ with xRy.

We say that x *satisfies* ϕ, or that ϕ is true in world x, if $x \models \phi$.

A model $\mathcal{M}$ satisfies ϕ, written $\mathcal{M} \models \phi$, if $x \models \phi$, for all worlds $x \in W$.

A modal logic formula ϕ is *valid*, written $\models \phi$, if it is true in every world of every model.

Equivalences Between Modal Formulas

We say that a set of modal logic formulas Γ *semantically entails* a formula ϕ, written $\Gamma \models \phi$, if for every model $\mathcal{M}$ and every world x of $\mathcal{M}$, we have $x \models \phi$ whenever $x \models \psi$ for all $\psi \in \Gamma$.

We say that ϕ and ψ are *semantically equivalent*, written $\phi \equiv \psi$, if $\phi \models \psi$ and $\psi \models \phi$.

The equivalences of propositional logic hold more generally for modal logic formulas.

For example, any two modal logic formulas of the form $\neg(\alpha \wedge \beta)$ and $\neg\alpha \vee \neg\beta$ are semantically equivalent.

Other equivalences directly depend on the model operators:

$$\begin{array}{rcl}
\neg\Box\phi & \equiv & \Diamond\neg\phi \\
\neg\Diamond\phi & \equiv & \Box\neg\phi \\
\Box(\phi \wedge \psi) & \equiv & \Box\phi \wedge \Box\psi \\
\Diamond(\phi \vee \psi) & \equiv & \Diamond\phi \vee \Diamond\psi \\
\Box\top & \equiv & \top \\
\Diamond\top & \not\equiv & \top \\
\Diamond\bot & \equiv & \bot \\
\Box\bot & \not\equiv & \bot \\
\Diamond\top & \equiv & \Box p \rightarrow \Diamond p
\end{array}$$

Examples of Modal Logic Formulas

An important formula scheme, called K, that can easily be shown to be valid is the following:

$$\Box(\phi \rightarrow \psi) \land \Box\phi \rightarrow \Box\psi.$$

This formula scheme is often written in equivalent form as

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi).$$

Many non-valid formulas still make sense for certain interpretations of the model operators. These include:

$$\begin{aligned} T: &\quad \Box\phi \rightarrow \phi && (1)\\ 4: &\quad \Box\phi \rightarrow \Box\Box\phi && (2)\\ 5: &\quad \Diamond\phi \rightarrow \Box\Diamond\phi && (3)\\ D0: &\quad \Diamond\top && (4)\\ B: &\quad \Box\phi \rightarrow \Diamond\phi && (5) \end{aligned}$$

Different Readings of the Modal Operators

Basic modal logic can be refined in various ways, by assuming (non-logical) axioms and/or properties of the accessibility relation, so as to capture specific readings of the modal operators.

Necessity

$\Box\phi$: The formula ϕ is necessarily true.
$\Diamond\phi$: The formula ϕ is possibly true.

For this reading axioms 1 to 5 are assumed to be true.

Belief

$\Box\phi$: I believe that ϕ.
$\Diamond\phi$: ϕ is consistent with my beliefs.

For this reading axioms 2 to 5 are assumed to be true.

Knowledge

$\Box\phi$: I know that ϕ.
$\Diamond\phi$: For all I know, ϕ.

For this reading axioms 1 to 5 are assumed to be true.

Moral Obligation

$\Box\phi$: It ought to be that ϕ.
$\Diamond\phi$: It is permitted to be that ϕ.

For this reading axioms 4 and 5 are assumed to be true.

Properties of the Accessibility Relation

In most applications of modal logic one assumes that the accessibility relation satisfies certain properties.

For instance, R may be required to be:

- *reflexive*: xRx, for all $x \in W$;
- *transitive*: xRz whenever xRy and yRz;
- *symmetric*: yRx whenever xRy;
- *serial*: for every x there is a y such that xRy;
- *Euclidian*: yRz whenever xRy and xRz;
- *functional*: for every x there is a *unique* y such that xRy;
- *linear*: if xRy and xRz, then yRz or zRy or $y = z$.

Correspondence Theory

There is a connection between formula schemes and properties of the accessibility relation, which we will next explore.

Definition.

> A *frame* is a pair (W, R), where R is a binary relation on the set W (i.e., $R \subseteq W \times W$).

In other words, a frame is a Kripke model without a labelling function.

Definition.

> We say that a frame $\mathcal{F} = (W, R)$ *satisfies* a formula ϕ, written $\mathcal{F} \models \phi$, if for each Kripke model $\mathcal{M} = (W, R, L)$ and each world $x \in W$, we have $\mathcal{M}, x \models \phi$.

Correspondence Theory (cont.)

Theorem.

Let $\mathcal{F}$ be a frame (W, R). Then

1. R is reflexive iff $\mathcal{F} \models \Box\phi \to \phi$.
2. R is symmetric iff $\mathcal{F} \models \phi \to \Box\Diamond\phi$.
3. R is transitive iff $\mathcal{F} \models \Box\phi \to \Box\Box\phi$.
4. R is serial iff $\mathcal{F} \models \Box\phi \to \Diamond\phi$.
5. R is Euclidian iff $\mathcal{F} \models \Diamond\phi \to \Box\Diamond\phi$.
6. R is linear iff

$$\mathcal{F} \models \Box(\phi \land \Box\phi \to \psi) \lor \Box(\psi \land \Box\psi \to \phi).$$

Let $\mathcal{F}$ be a frame (W, R). We prove the following.

1. If R is symmetric then $\mathcal{F} \models \phi \to \Box\Diamond\phi$.

   Suppose R is symmetric. We need to show that

   $$\mathcal{M}, x \models \phi \to \Box\Diamond\phi$$

   for all Kripke models $\mathcal{M} = (W, R, L)$ and worlds $x \in W$.

   Take an arbitrary Kripke model $\mathcal{M}_0 = (W, R, L_0)$ and an arbitrary world $x_0 \in W$. Let us assume $\mathcal{M}_0, x_0 \models \phi$. We need to show $\mathcal{M}_0, x_0 \models \Box\Diamond\phi$.

   If $y_0 \in W$ is a world such that $x_0 R y_0$, then by the symmetry of R we also have $y_0 R x_0$. Since $x_0 \models \phi$ we may infer that $\mathcal{M}_0, y_0 \models \Diamond\phi$, for all worlds y_0 with $x_0 R y_0$. Thus we may conclude that $\mathcal{M}_0, x_0 \models \Box\Diamond\phi$, which completes the proof.

2. If $\mathcal{F} \models p \to \Box\Diamond p$ then R is symmetric.

   We prove this by contradiction.

   Suppose $\mathcal{F}$ is a frame such that $\mathcal{F} \models p \to \Box\Diamond p$, but R is not symmetric. If R is not symmetric there exist $x_0 \in W$ and $y_0 \in W$ such that $x_0 R y_0$, but not $y_0 R x_0$.

   We know that

   $$(W, R, L), x \models p \to \Box\Diamond p$$

for all labelling functions L and all worlds $x \in W$.

Let L_0 be a labelling function such that (i) $p \in L_0(x_0)$ and (ii) $p \notin L_0(x)$ if $x \neq x_0$. Let $\mathcal{M}_0$ be the Kripke model (W, R, L_0).

We have $\mathcal{M}_0, x_0 \models p$ and $\mathcal{M}_0, x_0 \models p \rightarrow \Box\Diamond p$, from which we may infer $\mathcal{M}_0, x_0 \models \Box\Diamond p$.

From the latter fact, and $x_0 R y_0$, we obtain $\mathcal{M}_0, y_0 \models \Diamond p$. Thus there must be a world z such that $y_0 R z$ and $\mathcal{M}_0, z \models p$.

By the definition of L_0, the only possible candidate for z is x_0. But then $y_0 R x_0$, which is a contradiction.