Modal Logics

Most applications of modal logic require a refined version of basic modal logic.

Definition. A set L of formulas of basic modal logic is called a *(normal) modal logic* if the following closure properties are satisfied:

1. *Closure with respect to propositional logic*: The set L contains all formulas that can be derived from it by propositional logic reasoning.

2. *Closure with respect to K*: The set L contains all instances of the formula scheme K.

3. *Closure under necessitation*: If $\phi \in L$, then $\Box\phi \in L$.

4. *Closure under instantiation*: If $\phi \in L$, then L also contains every formula that is a substitution instance of ϕ.

Specific modal logics are specified by giving formula schemes, which are then called axioms, and considering the corresponding closure as defined above. Note that by definition, K is an axiom of any such logic.

The Modal Logics S4 and S5

The logic S4, or KT4, is characterized by axioms T and 4 (and K).

The logic S5, or KT45, is characterized by axioms T, 4, and 5 (and K).

The latter logic is used to reason about knowledge and $\Box\phi$ is read as "I know ϕ."

The axioms then correspond to the following assumptions:

T. Truth: I know only true things.

4. Positive introspection: If I know something, then I know that I know it.

5. Negative introspection: If I don't know something, then I know that I don't know it.

Also note that axiom K expresses logical omniscience in that it expresses that one's knowledge is closed under logical consequence.

The logic $S5$ thus formalizes an idealized concept of knowledge, different from human knowledge.

Semantic Entailment

Let L be a modal logic.

We say that a set Γ of basic modal logic formulas semantically entails ϕ *in* L and write

$$\Gamma \models_L \phi$$

if $\Gamma \cup L$ semantically entails ϕ (in basic modal logic), i.e., for every model $\mathcal{M}$ and every world x of $\mathcal{M}$, such that $x \models \psi$ for all $\psi \in \Gamma \cup L$, we also have $x \models \phi$.

In general it is difficult to determine the semantic entailment relation based on its definition, but the relation can also be characterized proof-theoretically via natural deduction.

The Wise-Men Puzzle

There are three wise men, three red hats, and two white hats. The king puts a hat on each of the wise men in such a way that they are not able to see their own hat. He then asks each one in turn whether he knows the color of his hat.

The first man says he does not know. The second man says he does not know either.

What does the third man say?

The Muddy-Children Puzzle

There is a variation on the wise-men puzzle in which the questions are not asked sequentially, but in parallel.

A group of children is playing in the garden and some of the children, say k of them, get mud on their foreheads. Each child can see the mud on others only. Note that if $k > 1$, then every child can see another with mud on its forehead.

Now consider two scenarios:

1. The father repeatedly asks "Does any of you know whether you have mud on your forehead?". All children answer "no" the first time, and continue to answer "no" to repetitions of the same question.

2. The father tells the children that at least one of them is muddy and repeatedly asks "Does any of you know whether you have mud on your forehead?". This time, after the question has been asked k times, the k muddy children will answer "yes."

The Muddy-Children Puzzle (cont.)

Consider the second scenario.

$k = 1$. There is only one muddy child, which will answer "yes" because of the father's statement.

$k = 2$. If two children, call them a and b, are muddy, they both answer "no" the first time. But both a and b then reason that the other muddy child must have seen someone with mud, and hence answers "yes" the second time.

$k = 3$. Let a, b, and c be the muddy children. Everybody answers "no" the first two times. But then a reasons that if b and c are the only muddy children they would have answered "yes" the second time (based on the argument for the case $k = 2$). Since they answered "no," a further reasons, they must have seen a third child with mud, which must be me. Children b and c reason in the same way, and all three children answer "yes" the third time.

Note that the father's announcement makes it *common knowledge* among the children that at least one child is muddy.

The Modal Logic $KT45^n$

Many applications require models with several interacting agents. We next generalize the logic $KT45$ by introducing multiple versions of the $\Box$ operator as well as new modal operators.

Let $\mathcal{A}$ be a set, the elements of which are called *agents*.

The language of the modal logic $KT45^n$ is defined by the following syntax rules:

$$\begin{aligned}\phi \quad ::= \quad & \top \mid \bot \mid P \mid (\neg\phi) \\ & \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\ & \mid K_i\phi \mid E_G\phi \mid C_G\phi \mid D_G\phi\end{aligned}$$

where P is any atomic formula, $i \in \mathcal{A}$, and $G \subseteq \mathcal{A}$.

We write E, C, and D without subscripts if $G = \mathcal{A}$.

Semantics of $KT45^n$

A *model* $\mathcal{M}$ of the multi-modal logic $KT45^n$ is a triple $(W, (R_i)_{i \in \mathcal{A}}, L)$, where

1. W is a (non-empty) set,
2. R_i is an equivalence relation on W, for each $i \in \mathcal{A}$, and
3. L is a function from W to $\mathcal{P}(A)$, where A is the given set of atoms.

Let $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ be a model of $KT45^n$. We use structural induction to define the following relation, for $x \in W$ and multi-modal formulas ϕ:

- $x \models K_i\phi$ iff $y \models \phi$, for each $y \in W$ with xR_iy.
- $x \models E_G\phi$ iff $x \models K_i\phi$, for each $i \in G$.
- $x \models C_G\phi$ iff $x \models E_G^k\phi$, for all natural numbers $k \geq 1$. (Here $E_G^{k+1} = E_G E_G^k$ and $E_G^1 = E_G$.)
- $x \models D_G\phi$ iff $y \models \phi$, for all $y \in W$ such that xR_iy, for all $i \in G$.

(The clauses for the propositional connectives are similar as in the definition of other satisfaction relations.)

Formalizing the Wise-Men Puzzle

Let p_i mean that man i wears a red hat and $\neg p_i$ mean that he wears a white hat.

Let Γ be the set of formulas

$$\begin{aligned}
&\{C(p_1 \vee p_2 \vee p_3),\\
&C(p_1 \to K_2 p_1),\ C(\neg p_1 \to K_2 \neg p_1),\\
&C(p_1 \to K_3 p_1),\ C(\neg p_1 \to K_3 \neg p_1),\\
&C(p_2 \to K_1 p_2),\ C(\neg p_2 \to K_1 \neg p_2),\\
&C(p_2 \to K_3 p_2),\ C(\neg p_2 \to K_3 \neg p_2),\\
&C(p_3 \to K_1 p_3),\ C(\neg p_3 \to K_1 \neg p_3),\\
&C(p_3 \to K_2 p_3),\ C(\neg p_3 \to K_2 \neg p_3)\}.
\end{aligned}$$

The solution to the puzzle rests on two observations:

1. The set of formulas Γ plus $C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1)$ semantically entails (with respect to $KT45^n$) $C(p_2 \vee p_3)$.

2. The set of formulas Γ plus $C(p_2 \vee p_3)$ and $C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2)$ semantically entails $K_3 p_3$.

Formalizing the Muddy-Children Puzzle

Let p_i mean that child i is muddy.

Let Γ be the set of all formulas of the form

$$C(p_1 \vee p_2 \vee \cdots \vee p_n)$$

or

$$\bigwedge_{i \neq j} C(p_i \rightarrow K_j p_i)$$

or

$$\bigwedge_{i \neq j} C(\neg p_i \rightarrow K_j \neg p_i).$$

Furthermore, let α_G denote the formula

$$\bigwedge_{i \in G} p_i \wedge \bigwedge_{i \notin G} \neg p_i.$$

The following can be proved:

1. The set of formulas Γ plus $\alpha_{\{i\}}$ semantically entails $K_i p_i$.

2. The set of formulas Γ plus $C(\bigwedge_{|G| \leq k} \neg \alpha_G)$ and α_H semantically entails $\bigwedge_{i \in H} K_i p_i$, where $|H| = k + 1$.