

**CSE541      EXERCISE 10 SOLUTIONS**

**Covers Chapters 10, 11, 12** Read and learn all examples and exercises in the chapters as well!

**QUESTION 1**

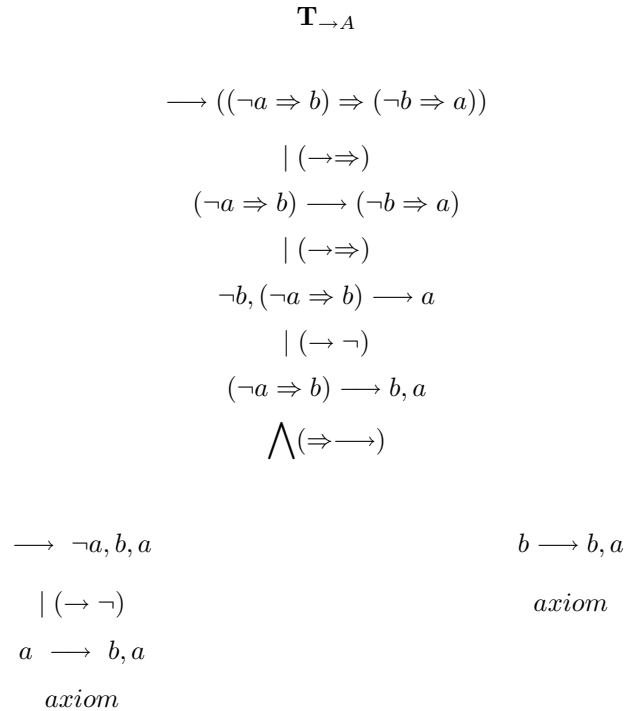
Let **GL** be the Gentzen style proof system for classical logic defined in chapter 11. Prove, by constructing a proper decomposition tree that

(1)  $\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$ .

**Solution:** By definition we have that

$$\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \text{ if and only if } \vdash_{\mathbf{GL}} \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

We construct the decomposition tree for  $\longrightarrow A$  as follows.



All leaves of the tree  $\mathbf{T}_{\longrightarrow A}$  are axioms, hence we have found the proof of  $A$  in **GL**.

(2) Let **GL** be the Gentzen style proof system defined in chapter 11. Prove, by constructing a proper decomposition tree that

$$\not\vdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

**Solution:** Observe that for any formula  $A$ , its decomposition tree  $\mathbf{T}_{\rightarrow A}$  in **GL** is not unique. Hence when constructing decomposition trees we have to cover all possible cases.

We construct the decomposition tree for  $\rightarrow A$  as follows.

$$\begin{array}{c}
 \mathbf{T}_{\mathbf{1} \rightarrow A} \\
 \\
 \rightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
 | (\rightarrow \Rightarrow) \\
 \text{(one choice)} \\
 (a \Rightarrow b) \rightarrow (\neg b \Rightarrow a) \\
 | (\rightarrow \Rightarrow) \\
 \text{(first of two choices)} \\
 \neg b, (a \Rightarrow b) \rightarrow a \\
 | (\neg \rightarrow) \\
 \text{(one choice)} \\
 (a \Rightarrow b) \rightarrow b, a \\
 \bigwedge (\Rightarrow \rightarrow) \\
 \text{(one choice)} \\
 \\
 \rightarrow a, b, a \qquad \qquad \qquad b \rightarrow b, a \\
 \text{non - axiom} \qquad \qquad \qquad \text{axiom}
 \end{array}$$

The tree contains a non- axiom leaf

$$\rightarrow a, b, a,$$

hence it is not a proof of

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

in **GL**. We have only one more tree to construct. Here it is.

$\mathbf{T}_{2 \rightarrow A}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$| (\rightarrow \Rightarrow)$$

*(one choice)*

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge (\Rightarrow \longrightarrow)$$

*(second of two choices)*

$$\longrightarrow (\neg b \Rightarrow a), a$$

$$(\longrightarrow \Rightarrow)$$

*(one choice)*

$$\neg b \longrightarrow a, a$$

$$| (\neg \rightarrow)$$

*(one choice)*

$$\longrightarrow b, a, a$$

*non - axiom*

$$b \longrightarrow (\neg b \Rightarrow a)$$

$$| (\rightarrow \Rightarrow)$$

*(one choice)*

$$b, \neg b \longrightarrow a$$

$$| (\neg \rightarrow)$$

*(one choice)*

$$b \longrightarrow b, a$$

*axiom*

All possible trees end with an non-axiom leaf whet proves that

$$\not\vdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

**QUESTION 2**

Does the tree below constitute a proof in **GL**? Justify your answer.

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow \neg \neg ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$| (\rightarrow \neg)$$

$$\neg ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\begin{array}{c}
| (\neg \rightarrow) \\
\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
| (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
| (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b), \neg b \longrightarrow a \\
| (\neg \rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow b, a \\
\bigwedge (\Rightarrow \longrightarrow) \\
\longrightarrow \neg a, b, a \qquad \qquad \qquad b \longrightarrow b, a \\
| (\rightarrow \neg) \qquad \qquad \qquad \text{axiom} \\
a \longrightarrow b, a \\
\text{axiom}
\end{array}$$

**Solution:** The tree is not a proof in **GL** because a rule corresponding to the decomposition step below does not exist in it.

$$\begin{array}{c}
(\neg a \Rightarrow b), \neg b \longrightarrow a \\
| (\neg \rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow b, a
\end{array}$$

It is a proof in some system **GL1** that has all the rules of **GL** except its  $(\neg \rightarrow)$ . This rule has to be replaced by the rule:

$$(\neg \rightarrow)_1 \frac{\Gamma, \Gamma' \longrightarrow \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \longrightarrow \Delta, \Delta'}.$$

Also the step above this one, i.e.

$$\begin{array}{c}
(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
| (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b), \neg b \longrightarrow a
\end{array}$$

is incorrect for similar reason.

**Observe** that the completeness of the system **GL** may not imply the completeness of **GL1**, i.e. we don't know if the new system **GL1** is complete (in fact, it is!).

**QUESTION 3**

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 10. Prove, by constructing a counter-model defined by a proper decomposition tree that

$$\not\models ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b))).$$

**Solution**

$$\begin{array}{c}
 \mathbf{T}_{\rightarrow A} \\
 \longrightarrow ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b))) \\
 | (\rightarrow \Rightarrow) \\
 (a \Rightarrow (\neg b \cap a)) \longrightarrow (\neg b \Rightarrow (a \cup b)) \\
 | (\rightarrow \Rightarrow) \\
 \text{one of two choices} \\
 \neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow (a \cup b) \\
 | (\rightarrow \cup) \\
 \text{one of two choices} \\
 \neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow a, b \\
 | (\neg \rightarrow) \\
 (a \Rightarrow (\neg b \cap a)) \longrightarrow b, a, b \\
 \bigwedge (\Rightarrow \longrightarrow) \\
 \longrightarrow a, b, a, b \qquad \longrightarrow (\neg b \cap a), b, a \\
 \text{non - axiom} \qquad \bigwedge (\longrightarrow \cap) \\
 \longrightarrow \neg b, b, a, b \qquad \longrightarrow a, b, a, b \\
 | (\rightarrow \neg) \qquad \text{non - axiom} \\
 b \longrightarrow b, a, b \\
 \text{axiom}
 \end{array}$$

The counter-model model determined by the non-axiom leaf  $\longrightarrow a, b, a, b$  is any truth assignment that evaluates it to  $F$ .

Observe that (we use a shorthand notation)  $\longrightarrow a, b, a, b$  represents semantically  $T \longrightarrow a, b, a, b$  and hence  $\longrightarrow a, b, a, b = F$  iff  $T \longrightarrow a, b, a, b = F$ , what happens only if  $T \Rightarrow a \cup b \cup a \cup b = F$ , i.e when  $a = F$  and  $b = F$ .

#### QUESTION 4

Consider a system **RS1** obtained from **RS** by changing the sequence  $\Gamma'$  into  $\Gamma$  and  $\Delta$  into  $\Delta'$  in all of the rules of inference of **RS**.

1. Construct a decomposition tree of

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

2. Define in your own words, for any  $A$ , the decomposition tree  $\mathbf{T}_A$  in **RS1**.
3. Prove **Completeness Theorem** for **RS1**.

1. Construct a decomposition tree of  $A$  from the QUESTION 1 in **RS1**.

#### Solution

$$\begin{array}{c}
 \mathbf{T}_A \\
 (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \\
 | (\Rightarrow) \\
 (\neg\neg(a \cap b), (\neg a \cup \neg b)) \\
 | (\cup) \\
 (\neg\neg(a \cap b), \neg a, \neg b) \\
 | (\neg\neg) \\
 ((a \cap b), \neg a, \neg b) \\
 \bigwedge (\cap) \\
 \\
 \begin{array}{cc}
 a, \neg a, \neg b & b, \neg a, \neg b \\
 axiom & axiom
 \end{array}
 \end{array}$$

2. Define in your own words, for any  $A$ , the decomposition tree  $\mathbf{T}_A$  in **RS1**.

**Solution** Steps are as follows.

1. Decompose using rule defined by the main connective of  $A$
2. Scan resulting sequence from RIGHT to LEFT and find first decomposable formula  $A$
3. Repeat 1. and 2. until no more decomposable formulas.
4. Tree  $\mathbf{T}_A$  is a proof if all leaves are axioms.
5. The proof does not exist otherwise, i.e. there is a non- axiom leaf because the tree is, as in **RS**, unique.

**3. Prove Completeness Theorem for RS1.**

**Assume** that  $\not\vdash_{\mathbf{RS1}} A$ . From 1-5 we have that there is a leaf  $L$  of the decomposition tree of  $A$ , which is not an axiom.

**Observe**, that **RS1** is sound by the same proof as for **RS**. By soundness, if one premiss of a rule of **RS** is FALSE, so is the conclusion.

**Hence** by soundness and the definition of the decomposition tree any truth assignment  $v$  that falsifies an non axiom leaf, i.e. any  $v$  such that  $v^*(L) = F$  falsifies  $A$ , namely  $v^*(A) = F$  and hence constitutes a counter model for  $A$ . This ends that proof that  $\not\vdash A$ .

**Essential part:**

**Given a formula**  $A$  such that  $\not\vdash_{\mathbf{RS1}} A$  and its decomposition tree of  $A$  with a non-axiom leaf  $L$ .

We define a **counter-model**  $v$  determined by the non- axiom leaf  $L$  as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L \\ T & \text{if } \neg a \text{ appears in } L \\ \text{any value} & \text{if } a \text{ does not appear in } L \end{cases}$$

### QUESTION 5

Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 12.

Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

**Solution:** Observe that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad \text{iff} \quad \vdash_{\mathbf{LI}} \longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad .$$

Consider the following decomposition tree  $\mathbf{T}_{\rightarrow A}$  of  $\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$  in **LI**.

$$\begin{array}{c} \mathbf{T}_{\rightarrow A} \\ \longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\ | (\rightarrow \neg) \\ \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \\ | (\text{contr } \rightarrow) \end{array}$$

$$\begin{aligned}
& \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \\
& \quad | (\neg \rightarrow) \\
& \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
& \quad | (\rightarrow \Rightarrow) \\
& (\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow (\neg b \Rightarrow a) \\
& \quad | (\rightarrow \Rightarrow) \\
& \neg b, (\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a \\
& \quad | (exch \rightarrow) \\
& (\neg a \Rightarrow b), \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a \\
& \quad \bigwedge (\Rightarrow \rightarrow)
\end{aligned}$$

$$\begin{array}{ll}
\neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \neg a & b, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \\
\quad | (\rightarrow \neg) & \quad | (exch \rightarrow) \\
a, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow & \neg b, b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \\
\quad | (exch \rightarrow) & \quad | (\neg \rightarrow) \\
a, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg b \longrightarrow & b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow b \\
\quad | (exch \rightarrow) & \quad \text{axiom} \\
\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), a, \neg b \longrightarrow & \\
\quad | (\neg \rightarrow) & \\
a, \neg b \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) & \\
\quad | (\rightarrow \Rightarrow) & \\
(\neg a \Rightarrow b), a, \neg b \longrightarrow (\neg b \Rightarrow a) & \\
\quad | (\rightarrow \Rightarrow) & \\
\neg b, (\neg a \Rightarrow b), a, \neg b \longrightarrow a & \\
\quad \text{axiom} &
\end{array}$$

All leaves of  $\mathbf{T}_{\rightarrow, A}$  are axioms, we have hence found a proof.

### QUESTION 5

We know that the formulas below are not Intuitionistic Tautologies. Verify whether  $\mathbf{H}$  semantics (chapter 5) provides a counter-model for them.

$$\begin{aligned}
& ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \\
& ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))
\end{aligned}$$

### Solution

First Formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

We evaluate:  $((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = \perp$  iff  $(a \Rightarrow b) = T$  and  $(\neg a \cup b) = \perp$ . Observe that  $(\neg a \cup b) = \perp$  in 3 cases, two of which for  $\neg a = \perp$  are impossible. We have hence only one case to consider:  $\neg a = F, b = \perp$ , i.e.  $a = \perp$  or  $a = T$  and  $b = \perp$ . Both of them provide a counter-model.

item[] Second formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

**Solution**  $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)) = \perp$  iff  $(\neg a \Rightarrow \neg b) = T$  and  $(b \Rightarrow a) = \perp$ . The case  $(b \Rightarrow a) = \perp$  holds iff  $b = T$  and  $a = \perp$ . In this case  $(\neg a \Rightarrow \neg b) = (\neg \perp \Rightarrow \neg T) = F \Rightarrow F = T$ . We have a counter-model.

### QUESTION 6

Show that

$$\vdash_{\mathbf{LI}} \neg \neg ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

**Solution** We did work it out in chapter 12 and in2class.

**QUESTION 7** Use the heuristic method defined in chapter 12 to prove that

$$\not\vdash_{\mathbf{LI}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

**Solution:** To prove that our formula is not provable in **LI** we construct its possible decomposition trees following our heuristic, discuss their relationship and show that each of them must have an non-axiom leaf.

**First tree** is as follows.

$$\begin{array}{c}
\mathbf{T1} \\
\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
\quad | (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
\quad | (\rightarrow \Rightarrow) \\
\neg b, (\neg a \Rightarrow b) \longrightarrow a \\
\quad | (exch \rightarrow) \\
(\neg a \Rightarrow b), \neg b, \longrightarrow a \\
\quad \bigwedge (\Rightarrow \longrightarrow)
\end{array}$$

$\neg b \longrightarrow \neg a$	$b, \neg b \longrightarrow a$
( $\rightarrow \neg$ )	( <i>exch</i> $\rightarrow$ )
$a, \neg b \longrightarrow$	$\neg b, b \longrightarrow a$
( <i>exch</i> $\rightarrow$ )	( $\rightarrow$ <i>weak</i> )
$\neg b, a \longrightarrow$	$\neg b, b \longrightarrow$
( $\neg \rightarrow$ )	( $\neg \rightarrow$ )
$a \longrightarrow b$	$b \longrightarrow b$
<i>non - axiom</i>	<i>axiom</i>

**Second tree** The second choice of decomposition rule at the second node of the tree **T1** gives the following tree.

<b>T2</b>	
$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$	
( $\rightarrow \Rightarrow$ )	
$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$	
$\bigwedge (\Rightarrow \longrightarrow)$	
$\longrightarrow \neg a$	$b \longrightarrow (\neg b \Rightarrow a)$
( $\rightarrow \neg$ )	( $\rightarrow \Rightarrow$ )
$a \longrightarrow$	$b, \neg b \longrightarrow a$
<i>non - axiom</i>	( <i>exch</i> $\rightarrow$ )
	$\neg b, b \longrightarrow a$
	( $\rightarrow$ <i>weak</i> )
	$\neg b, b \longrightarrow$
	( $\neg \rightarrow$ )
	$b \longrightarrow b$
	<i>axiom</i>

**Observe** that **T1** and **T2** have identical sub-trees ending with identical leaves.

**Third tree** is obtained by the third choice of the decomposition rule at the second node of the tree **T1**, namely the use of rule (*contr*  $\rightarrow$ ). This step produces a node

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

**Observe** that next decomposition steps would give trees similar to **T1** and **T2**. We write down, as an example one of them, which follows the pattern of the tree **T1**.

**T3**

$$\begin{aligned}
& \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
& \quad | (\rightarrow \Rightarrow) \\
& (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
& \quad | (contr \rightarrow) \\
& (\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
& \quad | (\rightarrow \Rightarrow) \\
& \neg b, (\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow a \\
& \quad | (exch \rightarrow) \\
& (\neg a \Rightarrow b), \neg b, (\neg a \Rightarrow b) \longrightarrow a \\
& \quad \bigwedge (\Rightarrow \longrightarrow)
\end{aligned}$$

$ \begin{aligned} & \neg b, (\neg a \Rightarrow b) \longrightarrow \neg a \\ & \quad   (\rightarrow \neg) \\ & a, \neg b, (\neg a \Rightarrow b) \longrightarrow \\ & \quad   (exch \rightarrow) \\ & \neg b, a, (\neg a \Rightarrow b) \longrightarrow \\ & \quad   (\neg \rightarrow) \\ & a, (\neg a \Rightarrow b) \longrightarrow b \\ & \quad   (exch \rightarrow) \\ & (\neg a \Rightarrow b), a \longrightarrow b \\ & \quad \bigwedge (\Rightarrow \longrightarrow) \end{aligned} $	$ \begin{aligned} & \bigwedge (\Rightarrow \longrightarrow) \\ & b, \neg b, (\neg a \Rightarrow b) \longrightarrow a \\ & \quad   (exch \rightarrow) \\ & \neg b, b, (\neg a \Rightarrow b) \longrightarrow a \\ & \quad   (\rightarrow weak) \\ & \neg b, b, (\neg a \Rightarrow b) \longrightarrow \\ & \quad   (\neg \rightarrow) \\ & b, (\neg a \Rightarrow b) \longrightarrow b \\ & \quad axiom \end{aligned} $
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$ \begin{aligned} & a \longrightarrow \neg a \\ & \quad   (\rightarrow \neg) \\ & a, a \longrightarrow \\ & non - axiom \end{aligned} $	$ \begin{aligned} & b, a \longrightarrow b \\ & \quad axiom \end{aligned} $
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**Observe** that the rule (*contr*  $\rightarrow$ ) didn't and will never bring information to the tree construction which would replace a non-axiom leaf by an axiom leaf.

**Next tree** can be obtained by exploring second choice at the node 3 of the first tree.

$$\begin{array}{c}
 \mathbf{T4} \\
 \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
 | (\rightarrow \Rightarrow) \\
 (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
 | (\rightarrow \Rightarrow) \\
 \neg b, (\neg a \Rightarrow b) \longrightarrow a \\
 | (\rightarrow \textit{weak}) \\
 \neg b, (\neg a \Rightarrow b) \longrightarrow \\
 | (\neg \rightarrow) \\
 (\neg a \Rightarrow b) \longrightarrow b \\
 \bigwedge (\Rightarrow \rightarrow) \\
 \\
 \begin{array}{ccc}
 \longrightarrow \neg a & & b \longrightarrow b \\
 | (\rightarrow \neg) & & \textit{axiom} \\
 a \longrightarrow & & \\
 \textit{non - axiom} & & 
 \end{array}
 \end{array}$$

**Observe** that here again the rule (*contr*  $\rightarrow$ ) applied to any node to the tree **T4** would never gives us a possibility of replacing a non-axiom leaf by an axiom leaf.

**Conclusion** All possible decomposition trees will always contain a non- axiom leaf what ends the proof.

**GENERAL REMARK** We are using the word "PROOF" in two distinct senses.

In the first sense, we use it as a *formal proof* within a fixed proof system, namely **LI** and is represented as a proof tree, or sequence of expressions of the language  $\mathcal{L}$  of **LI**.

In the second sense, it also designates certain sequences of sentences of English language (supplemented by some technical terms, if needed) that are supposed to serve as an argument justifying some assertions about the language  $\mathcal{L}$ , or proof system based on it.

In general, the language we are studying, in this case  $\mathcal{L}$ , is called *an OBJECT LANGUAGE*.

The language in which we formulate and prove the results about the object language is called *the META-LANGUAGE*. The metalanguage might also be formalized and made the object of study, which we would carry in *a meta-metalanguage*.

We use English as our not formalized metalanguage, although, we use only a mathematically weak portion of the English language. The contrast between the language and metalanguage is also present in study for example, a foreign language. In French study class, French is the object language, while the metalanguage, the language we use, is English.

The distinction between *proof* and *meta-proof*, i.e. a proof in the metalanguage, is now clear. We construct (in the metalanguage) a decomposition tree which is *a formal proof* in the object language. By doing so, we prove in the metalanguage, that the proof in the object language exists. Such proof is called *a meta-proof*, and the fact thus proved is called a *meta-theorem*.