CSE541 EXERCISE 12

Chapters 10, 11, 12 Read and learn all examples and exercises in the chapters as well!

QUESTION 1

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 11. Prove, by constructing a proper decomposition tree that

(1) $\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

Solution: By definition we have that

$$\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \ \ if\ and\ only\ if\ \ \vdash_{\mathbf{GL}} \ \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

We construct the decomposition tree for $\longrightarrow A$ as follows.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow \neg)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$\longrightarrow \ \neg a, b, a$	$b \longrightarrow b, a$
$\mid (\rightarrow \neg)$	*axiom*
$a \ \longrightarrow \ b, a$	
axiom	

All leaves of the tree $\mathbf{T}_{\rightarrow A}$ are axioms, hence we have found the proof of A in **GL**.

(2) Let **GL** be the Gentzen style proof system defined in chapter 11. Prove, by constructing a proper decomposition tree that

$$\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: Observe that for any formula A, its decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** is not unique. Hence when constructing decomposition trees we have to cover all possible cases.

We construct the decomposition tree for $\longrightarrow A$ as follows.

$$\mathbf{T_1}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$(first\ of\ two\ choices)$$

$$\neg b, (a \Rightarrow b) \ \longrightarrow \ a$$

$$\mid (\neg \rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ b, a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$(one\ choice)$$

$\longrightarrow \ a, b, a$	$b \longrightarrow b, a$
$non - axiom$	$axiom$

The tree contains a non- axiom leaf $\longrightarrow \ a, b, a$, hence it is not a proof of $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$ in **GL**. We have only one more tree to construct. Here it is.

$$\mathbf{T_2}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$(second\ of\ two\ choices)$$

$\longrightarrow \ (\neg b \Rightarrow a), a$	$b \longrightarrow \ (\neg b \Rightarrow a)$
$(\longrightarrow\Rightarrow)$	$\mid (\rightarrow\Rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\neg b \longrightarrow \ a, a$	$b, \neg b \longrightarrow \ a$
$\mid (\neg \rightarrow)$	$\mid (\neg \rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\longrightarrow \ b, a, a$	$b \ \longrightarrow \ b, a$
$non - axiom$	$axiom$

All possible trees end with an non-axiom leave whet proves that $\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

QUESTION 2

Show that tree below do not constitute a proof in **GL** defined in chapter 10? Provide a correct proof.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \neg)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a, b, a \qquad\qquad b \longrightarrow b, a$$

$$\mid (\rightarrow \neg) \qquad\qquad axiom$$

$$a \longrightarrow b, a$$

$$axiom$$

Solution: The tree is a not a proof in **GL** because a rule corresponding to the decomposition step below does not exists in it.

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

It is a proof is some system **GL1** that has all the rules of **GL** except its $(\neg \rightarrow)$. This rule has to be replaced by the rule:

$$(\neg \rightarrow)_1 \ \frac{\Gamma, \Gamma^{'} \ \longrightarrow \ \Delta, A, \Delta^{'}}{\Gamma, \neg A, \Gamma^{'} \ \longrightarrow \ \Delta, \Delta^{'}}.$$

Observe that the completeness of the system **GL** may not imply the completeness of **GL1**, i.e. we don't know if the new system **GL1** is complete (in fact, it is!).

QUESTION 3

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 11. Prove, by constructing a counter-model defined by a proper decomposition tree that

$$\not\models ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b))).$$

Solution

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow (\neg b \Rightarrow (a \cup b))$$

$$| (\rightarrow\Rightarrow))$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow (a \cup b))$$

$$| (\rightarrow \cup)$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow a, b$$

$$| (\neg \rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow b, a, b$$

$$\bigwedge (\Rightarrow \longrightarrow)$$

$$\longrightarrow \neg a, b, a, b \qquad\qquad \longrightarrow (\neg b \cap a), b, a$$

$$non - axiom \qquad\qquad \bigwedge (\longrightarrow \cap)$$

$$\longrightarrow \neg b, b, a, b \qquad\qquad \longrightarrow a, b, a, b$$

$$| (\rightarrow \neg) \qquad\qquad non - axiom$$

$$b \longrightarrow b, a, b$$

$$axiom$$

The counter-model model determined by the non-axiom leaf $\longrightarrow a, b, a, b$ is any truth assignment that evaluates it to F.

Observe that (we use a shorthand notation) $\longrightarrow a, b, a, b$ represents semantically $T \longrightarrow a, b, a, b$ and hence $\longrightarrow a, b, a, b = F$ iff $T \longrightarrow a, b, a, b = F$, what happens only if $T \Rightarrow a \cup b \cup a \cup b = F$, i.e when $a = F$ and $b = F$.

QUESTION 4 Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 11. Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

QUESTION 5 We know that the formulas below are not Intuitionistic Tautologies. Verify whether **H** semantics (chapter 5) provides a counter-model for them.

$$((a \Rightarrow b) \Rightarrow (\neg a \cup b)),$$

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)).$$

Solution

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow (\neg b \Rightarrow (a \cup b))$$

$$\mid (\rightarrow\Rightarrow))$$

one of two choices

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow (a \cup b))$$

$$\mid (\rightarrow \cup)$$

one of two choices

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow a, b$$

$$\mid (\neg \rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow b, a, b$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a, b, a, b$$

non − axiom

$$\longrightarrow (\neg b \cap a), b, a$$

$$\bigwedge(\longrightarrow \cap)$$

$$\longrightarrow \neg b, b, a, b$$

$$\mid (\rightarrow \neg)$$

$$b \longrightarrow b, a, b$$

axiom

$$\longrightarrow a, b, a, b$$

non − axiom

The counter-model model determined by the non-axiom leaf $\longrightarrow a, b, a, b$ is any truth assignment that evaluates it to F.

Observe that (we use a shorthand notation) $\longrightarrow a, b, a, b$ represents semantically $T \longrightarrow a, b, a, b$ and hence $\longrightarrow a, b, a, b = F$ iff $T \longrightarrow a, b, a, b = F$, what happens only if $T \Rightarrow a \cup b \cup a \cup b = F$, i.e when $a = F$ and $b = F$.

QUESTION 4 Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 11. Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: Observe that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad iff \quad \vdash_{\mathbf{LI}} \quad \longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad .$$

Consider the following decomposition tree $\mathbf{T}_{\rightarrow A}$ of $\rightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$ in **LI**.

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \neg)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (contr \rightarrow)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$a, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg b \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), a, \neg b \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a, \neg b \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(\neg a \Rightarrow b), a, \neg b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow \Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), a, \neg b \longrightarrow a$$

$$axiom$$

$$b, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg b, b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow b$$

$$axiom$$

All leaves of $\mathbf{T}_{\rightarrow A}$ are axioms, we have hence found a proof.

QUESTION 5 We know that the formulas below are not Intuitionistic Tautologies. Verify whether **H** semantics (chapter 5) provides a counter-model for them.

$$((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution

First Formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

We evaluate:$((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = \bot$ iff $(a \Rightarrow b) = T$ and $(\neg a \cup b) = \bot$. Observe that $(\neg a \cup b) = \bot$ in 3 cases, two of which for $\neg a = \bot$ are impossible. We have hence only one case to consider: $\neg a = F, b = \bot$, i.e. $a = \bot$ or $a = T$ and $b = \bot$. Both of them provide a counter-model.

item[] Second formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)) = \bot$ iff $(\neg a \Rightarrow \neg b) = T$ and $(b \Rightarrow a) = \bot$. The case $(b \Rightarrow a) = \bot$ holds iff $b = T$ and $a = \bot$. In this case $(\neg a \Rightarrow \neg b) = (\neg \bot \Rightarrow \neg T) = F \Rightarrow F = T$. We have a counter-model.

QUESTION 6 Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution in chapter 12.

QUESTION 7 Use the heuristic method defined in chapter 12 to prove that

$$\nvdash_{\mathbf{LI}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: To prove that our formula is not provable in **LI** we construct its possible decomposition trees following our heuristic, discuss their relationship and show that each of them must have an non-axiom leaf.

First tree is as follows.

T1

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow\Rightarrow)$$
$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\rightarrow\Rightarrow)$$
$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$
$$\mid (exch \rightarrow)$$
$$(\neg a \Rightarrow b), \neg b, \longrightarrow a$$
$$\bigwedge(\Rightarrow\longrightarrow)$$

$\neg b \longrightarrow \neg a$	$b, \neg b \longrightarrow a$
$\mid (\rightarrow \neg)$	$\mid (exch \rightarrow)$
$a, \neg b \longrightarrow$	$\neg b, b \longrightarrow a$
$\mid (exch \rightarrow)$	$\mid (\rightarrow weak)$
$\neg b, a \longrightarrow$	$\neg b, b \longrightarrow$
$\mid (\neg \rightarrow)$	$\mid (\neg \rightarrow)$
$a \longrightarrow b$	$b \longrightarrow b$
$non-axiom$	$axiom$

Second tree The second choice of decomposition rule at the second node of the tree **T1** gives the following tree.

T2

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a \longrightarrow$$

$$non-axiom$$

$$b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$b, \neg b \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$\neg b, b \longrightarrow a$$

$$\mid (\rightarrow weak)$$

$$\neg b, b \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b \longrightarrow b$$

$$axiom$$

Observe that **T1** and **T2** have identical sub-trees ending with identical leaves.

Third tree is obtained by the third choice of the decomposition rule at the second node of the tree **T1**, namely the use of rule $(contr \rightarrow)$. This step produces a node

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

Observe that next decomposition steps would give trees similar to **T1** and **T2**. We write down, as an example one of them, which follows the pattern of the tree **T1**.

T3

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (contr \rightarrow)$$

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), \neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$\neg b, (\neg a \Rightarrow b) \longrightarrow \neg a$

$\mid (\rightarrow \neg)$

$a, \neg b, (\neg a \Rightarrow b) \longrightarrow$

$\mid (exch \rightarrow)$

$\neg b, a, (\neg a \Rightarrow b) \longrightarrow$

$\mid (\neg \rightarrow)$

$a, (\neg a \Rightarrow b) \longrightarrow b$

$\mid (exch \rightarrow)$

$(\neg a \Rightarrow b), a \longrightarrow b$

$\bigwedge(\Rightarrow\longrightarrow)$

$\bigwedge(\Rightarrow\longrightarrow)$

$b, \neg b, (\neg a \Rightarrow b) \longrightarrow a$

$\mid (exch \rightarrow)$

$\neg b, b, (\neg a \Rightarrow b) \longrightarrow a$

$\mid (\rightarrow weak)$

$\neg b, b, (\neg a \Rightarrow b) \longrightarrow$

$\mid (\neg \rightarrow)$

$b, (\neg a \Rightarrow b) \longrightarrow b$

axiom

$a \longrightarrow \neg a$

$\mid (\rightarrow \neg)$

$a, a \longrightarrow$

non − axiom

$b, a \longrightarrow b$

axiom

Observe that the rule $(contr \rightarrow)$ didn't and will never bring information to the tree construction which would replace a non-axiom leaf by an axiom leaf.

Next tree can be obtained by exploring second choice at the node 3 of the first tree.

T4

$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

$\mid (\rightarrow\Rightarrow)$

$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$

$\mid (\rightarrow\Rightarrow)$

$\neg b, (\neg a \Rightarrow b) \longrightarrow a$

$\mid (\rightarrow weak)$

$\neg b, (\neg a \Rightarrow b) \longrightarrow$

$\mid (\neg \rightarrow)$

$(\neg a \Rightarrow b) \longrightarrow b$

$\bigwedge(\Rightarrow\longrightarrow)$

$$\longrightarrow \neg a \qquad\qquad\qquad\qquad b \longrightarrow b$$

$$\mid (\rightarrow \neg) \qquad\qquad\qquad\qquad axiom$$

$$a \longrightarrow$$

$$non - axiom$$

Observe that here again the rule ($contr \rightarrow$) applied to any node to the tree **T4** would never gives us a possibility of replacing a non-axiom leaf by an axiom leaf.

Conclusion All possible decomposition trees will always contain a non- axiom leaf what ends the proof.