CSE541 EXERCISE 1 SOLUTIONS

QUESTION 1 Describe a difference between logical and semantical paradoxes.

Logical paradoxes (antinomies) are paradoxes concerning the notion of a set.

Example: Russel paradox.

Consider the set A of all those sets X such that X is not a member of X. Clearly, by definition, A is a member of A if and only if A is not a member of A. So, if A is a member of A, the A is also not a member of A; and if A is not a member of A, then A is a member of A. In any case, A is a member of A and A is not a

Semantical paradoxes (antinomies) are paradoxes that deal with notion of truth, provability, and hence with logic.

Example: The Liar Paradox.

A man says: I am lying. *If he is lying, then what he says is true, and so he is not lying. If he is not lying, then what he says is not true, and so he is lying. In any case, he is lying and he is not lying.*

QUESTION 2

1. We translate our statement

From the fact that it is possible that Chris is not a boy we deduce that it is not possible that Chris is not a boy or, if it is possible that Chris is not a boy, then it is not necessary that Anne is pretty.

into a formula

(i) $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b.
a denotes statement: *Chris is a boy*, b denotes a statement: *Anne is pretty*.
Propositional Modal Connectives: $\mathbf{C}$, $\mathbf{I}$.
$\mathbf{C}$ denotes statement: *it is possible that*, $\mathbf{I}$ denotes statement: *it is necessary that*.

Translation 1:

$$A_1 = (\mathbf{C}\neg a \Rightarrow (\neg \mathbf{C}\neg a \cup (\mathbf{C}\neg a \Rightarrow \neg \mathbf{I} b))).$$

Now we translate our statement into a formula

(ii) $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b.

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a denotes statement: *it is possible that Chris is not a boy,*
b denotes a statement: *it is necessary that Anne is pretty.*

Translation 2:

$$A_2 = (a \Rightarrow (\neg a \cup (a \Rightarrow \neg b))).$$

2. Degree of the formula A_1 is: 12, degree of the formula A_2 is: 5.

3. All proper, non-atomic sub-formulas of A_1 are:

$$\mathbf{C}\neg a, \quad (\neg\mathbf{C}\neg a \cup (\mathbf{C}\neg a \Rightarrow \neg\mathbf{I}b), \quad (\mathbf{C}\neg a \Rightarrow \neg\mathbf{I}b), \quad \neg\mathbf{C}\neg a, \quad \neg\mathbf{I}b, \quad \mathbf{I}b, \quad \neg a$$

4. All proper, non-atomic sub-formulas of A_2 are:

$$(\neg a \cup (a \Rightarrow \neg b)), \quad (a \Rightarrow \neg b), \quad \neg a, \quad \neg b$$

5. Find a model and a counter-model restricted to A_2. Use short-hand notation. Don't construct Truth Tables!

Restricted Model: $a = F, b = F$.
Evaluation: $(F \Rightarrow (\neg F \cup (F \Rightarrow \neg F))) = (F \Rightarrow (T \cup (F \Rightarrow T))) = (F \Rightarrow (T \cup T)) = (F \Rightarrow T) = T$

Restricted Counter- Model: $a = T, b = T$.
Evaluation: $(T \Rightarrow (\neg T \cup (T \Rightarrow \neg T))) = (T \Rightarrow (F \cup (T \Rightarrow F))) = (T \Rightarrow (F \cup F)) = (T \Rightarrow F) = F$.

6. There is one possible counter-model restricted to A_2.

7. There are 3 possible models restricted to A_2. Justification: $2^2 - 1 = 3$.

8. List 3 models and 3 counter-models for A_2 by extending the model and the counter-model you have found in 4. to the VAR of all variables.

A model for A_2 is, by definition, any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = T$.

Model restricted to A_2 is, as defined in **5.** a function

$$v : \{a, b\} \longrightarrow \{T, F\},$$

such that $v(a) = F,\ v(b) = F$.

We extend v to the set of all propositional variables VAR to obtain a counter-model. Here are three of such extensions.

Model w_1:

$$w_1(a) = v(a), \quad w_1(b) = v(b) \quad and \; w_1(x) = T, \quad for \quad all \quad x \in VAR - \{a, b\}.$$

Model w_2:

$$w_2(a) = v(a), \; w_2(b) = v(b), \; w_2(c) = F \; and \; w_2(x) = T,$$
$$for \quad all \quad x \in VAR - \{a, b, c\}.$$

Model w_3:

$$w_3(a) = v(a), \; w_3(b) = v(b), \; w_3(c) = T \; and \; w_3(x) = T,$$
$$for \quad all \quad x \in VAR - \{a, b, c\}.$$

There is an many of such models, as extensions of v to the set VAR, i.e. as many as real numbers.

A counter-model for A_2, by definition, is any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = F$.

Counter- model restricted to A_2 is, as defined in **5.** a function

$$v : \{a, b\} \longrightarrow \{T, F\},$$

such that $v(a) = F, \; v(b) = F$.

There is only one **counter-model** v **restricted** to A_2.

We extend v to the set of all propositional variables VAR to obtain a counter-model. Here are three of such extensions.

Counter- model w_1:

$$w_1(a) = v(a) = T, \quad w_1(b) = v(b) = T \quad and \; w_1(x) = T, \quad for \quad all \quad x \in VAR - \{a, b\}.$$

Counter- model w_2:

$$w_2(a) = v(a) = T, \; w_2(b) = v(b) = T, \; w_2(c) = F \; and \; w_2(x) = T,$$
$$for \quad all \quad x \in VAR - \{a, b, c\}.$$

Counter- model w_3:

$$w_3(a) = v(a) = T, \; w_3(b) = v(b) = T, \; w_3(c) = T \; and \; w_3(x) = T,$$
$$for \quad all \quad x \in VAR - \{a, b, c\}.$$

There is an many of such counter- models, as extensions of v to the set VAR, i.e. as many as real numbers.

9. There are $2^{\aleph_0} = \mathcal{C}$ possible models for A_2. There are $2^{\aleph_0} = \mathcal{C}$ possible counter-models for A_2.

QUESTION 3 Show that

$$\models (\neg((a \cap \neg b) \Rightarrow ((c \Rightarrow (\neg f \cup d)) \cup e)) \Rightarrow ((a \cap \neg b) \cap (\neg(c \Rightarrow (\neg f \cup d)) \cap \neg e))).$$

Observe that $VAR_A = \{a, b, c, d, e, f\}$, so there are $2^6 = 64$ truth assignments to consider. Much too much to use the truth table method.

The "proof by contradiction" method may be shorter, but before we apply it let's look closer at the sub-formulas of A and patterns they form inside the formula A, i.e. we apply the substitution method first.
We denote : $B = (a \cap \neg b), C = (c \Rightarrow (\neg f \cup d)$, and $D = e$. We re-write A as

$$(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))).$$

Now we apply "proof by contradiction" method.

Step 1: Assume $(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))) = F$. It is possible **only** when $(B \Rightarrow (C \cup D)) = F$ and $(B \cap (\neg C \cap \neg D)) = F$.
Step 2: $(B \Rightarrow (C \cup D)) = F$ **only** when

$$B = T, C = F, D = F.$$

Step 3: From **Step 1** we have that

$$(B \cap (\neg C \cap \neg D)) = F.$$

We now evaluate its logical value for $B = T, C = F, D = F$ obtained in **Step 2**, i.e. compute:

$$(T \cap (\neg F \cap \neg F)) = F,$$

$$(T \cap (T \cap T)) = F,$$

$$T = F.$$

Contradiction. This proves that

$$\models (\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))),$$

and hence

$$\models (\neg((a \cup b) \Rightarrow ((c \Rightarrow d) \cup e)) \Rightarrow ((a \cup b) \cap (\neg(c \Rightarrow d) \cap \neg e))).$$

All truth assignments are models for A, i.e. A does not have a counter-model.