

CSE541 EXERCISE 3 SOLUTIONS

Reminder: We define **H** semantics operations \cup and \cap as follows

$$a \cup b = \max\{a, b\}, \quad a \cap b = \min\{a, b\}.$$

The **Truth Tables** for Implication and Negation are:

H-Implication

\Rightarrow	F	\perp	T
F	T	T	T
\perp	F	T	T
T	F	\perp	T

H Negation

\neg	F	\perp	T
	T	F	F

QUESTION 1 We know that

$$v : VAR \longrightarrow \{F, \perp, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$$

under **H** semantics. **evaluate** $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)))$.

Solution : $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under **H** semantics if and only if (we use shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ if and only if $a = T, b = T$ and $(T \Rightarrow c) = \perp$ if and only if $c = \perp$. I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp \quad \text{iff} \quad a = T, b = T, c = \perp.$$

Now we can we **evaluate** $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)))$ as follows (in shorthand notation).
 $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))) =$
 $((((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T))) =$
 $((T \Rightarrow (T \Rightarrow F)) \cup T) = T.$

We define a 4 valued \mathbf{L}_4 logic semantics as follows. The language is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of \mathbf{L}_4 as the following operations in the set $\{F, \perp_1, \perp_2, T\}$, where $\{F < \perp_1 < \perp_2 < T\}$.

Negation \neg

$$\neg : \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that

$$\neg \perp_1 = \perp_1, \quad \neg \perp_2 = \perp_2, \quad \neg F = T, \quad \neg T = F.$$

Conjunction \cap

$$\cap : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$,

$$a \cap b = \min\{a, b\}.$$

Disjunction \cup

$$\cup : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$,

$$a \cup b = \max\{a, b\}.$$

Implication \Rightarrow

$$\Rightarrow : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

QUESTION 2

Part 1 Write all TTables for \mathbf{L}_4

Solution :

\mathbf{L}_4 Negation

\neg	F	\perp_1	\perp_2	T
	T	\perp_1	\perp_2	F

\mathbf{L}_4 Disjunction

\cup	F	\perp_1	\perp_2	T
F	F	\perp_1	\perp_2	T
\perp_1	\perp_1	\perp_1	\perp_2	T
\perp_2	\perp_2	\perp_2	\perp_2	T
T	T	T	T	T

\mathbf{L}_4 Conjunction

\cap	F	\perp_1	\perp_2	T
F	F	F	F	F
\perp_1	F	\perp_1	\perp_1	\perp_1
\perp_2	F	\perp_1	\perp_2	\perp_2
T	F	\perp_1	\perp_2	T

\mathbf{L}_4 -Implication

\Rightarrow	F	\perp_1	\perp_2	T
F	T	T	T	T
\perp_1	\perp_1	T	T	T
\perp_2	\perp_2	\perp_2	T	T
T	F	\perp_1	\perp_2	T

Part 2 Verify whether

$$\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution : Let v be a truth assignment such that $v(a) = v(b) = \perp_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg \perp_1 \cup \perp_1)) = (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1$.

This proves that v is a counter-model for our formula and

$$\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a v such that $v(a) = v(b) = \perp_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg \perp_2 \cup \perp_2)) = (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2) = \perp_2$.

QUESTION 3 Prove using proper logical equivalences (list them at each step) that

1. $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$

Solution: $\neg(A \Leftrightarrow B) \stackrel{def}{\equiv} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \stackrel{deMorgan}{\equiv} \neg(A \Rightarrow B) \cup \neg(B \Rightarrow A)$
 $\stackrel{negimpl}{\equiv} ((A \cap \neg B) \cup (B \cap \neg A)) \stackrel{commut}{\equiv} ((A \cap \neg B) \cup (\neg A \cap B)).$

2. $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$

Solution: $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \stackrel{impl}{\equiv} \neg(B \cap \neg C) \cup (\neg A \cup B) \stackrel{deMorgan}{\equiv} (\neg B \cup \neg \neg C) \cup (\neg A \cup B)$
 $\stackrel{dneg}{\equiv} (\neg B \cup C) \cup (\neg A \cup B) \stackrel{impl}{\equiv} ((B \Rightarrow C) \cup (A \Rightarrow B)).$

QUESTION 4 We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

C1: For every formula A of \mathcal{L}_1 , there is a formula B of \mathcal{L}_2 , such that

$$A \equiv B,$$

C2: For every formula C of \mathcal{L}_2 , there is a formula D of \mathcal{L}_1 , such that

$$C \equiv D.$$

Prove that $\mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}.$

Solution: The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$