CSE541 EXERCISE 5 SOLUTIONS

QUESTION 1 Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \quad \mathcal{E} = \mathcal{F} \quad AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \quad (r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \quad).$$

Definition: System S is sound if and only if

(i) Axioms are tautologies and

(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that S is *sound* under classical semantics.

Solution:

(i) Both axioms of S are basic classical tautologies.

(ii) Consider the rule of inference of S.

$$(r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}.$$

Assume that its premise (the only premise) is True, i.e. let v be any truth assignment, such that $v^*(A \Rightarrow B) = T$. We evaluate logical value of the conclusion under the truth assignment v as follows.

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$.

2. Prove that S is *not sound* under **K** semantics.

Solution: Axiom $(A \Rightarrow A)$ is not a **K** semantics tautology.

QUESTION 2 Write a formal proof in S defined in Question 1 with 2 applications of the rule (r).

Solution: Required formal proof is a sequence $A_1, A_2, A_3,$ where
$A_1 = (A \Rightarrow A)$
(Axiom)
$A_2 = (A \Rightarrow (A \Rightarrow A))$
Rule (r) application 1 for $A = A,\ B = A$.
$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$.

QUESTION 3 Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))).$$

Solution: Required formal proof is a sequence $A_1, A_2,$ where

$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$

Axiom

$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$

Rule (r) application for $A = A, B = (\neg A \Rightarrow B)$.