CSE451 EXERCISE 6 SOLUTIONS

QUESTION 1 Given a proof system:

$$S = (\mathcal{L}_{\{\cup, \Rightarrow\}}, \quad \mathcal{E} = \mathcal{F} \quad AX = \{A1, A2\}, \quad \mathcal{R} = \{(r)\} \ ),$$

where

$$A1 = (A \Rightarrow (A \cup B)), \quad A2 = (A \Rightarrow (B \Rightarrow A))$$

and

$$(r) \ \frac{(A \Rightarrow B)}{(A \Rightarrow (A \Rightarrow B))}$$

1. Solution: Prove that S is *sound* under classical semantics.

Solution: Axioms of S are basic classical tautologies. The proof of soundness of the rule of inference is the following.

Assume $(A \Rightarrow B) = T$. Hence the logical value of conclusion is $(A \Rightarrow (A \Rightarrow B)) = (A \Rightarrow T) = T$ for all A.

2. Determine whether S is *sound* or *not sound* under **K** semantics.

K semantics differ from Łukasiewicz's semantics only in a case on implication only. This table is:

K-Implication

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	$\bot$	$\bot$	T
T	F	$\bot$	T

Solution 1: S is not sound under **K** semantics. Let's take truth assignment such that $A =\bot, B =\bot$. The logical value of axiom A1 is as follows.
$(A \Rightarrow (A \cup B)) = (\bot \Rightarrow (\bot \cup \bot)) = \bot$ and $\not\models_{\mathbf{K}} (A \Rightarrow (A \cup B))$.

Observe that the v such that $A =\bot, B =\bot$ is not the only v that makes $A1 \neq T$, i.e. proves that $\not\models_{\mathbf{K}} A1$.
$(A \Rightarrow (A \cup B)) \neq T$ if and only if $(A \Rightarrow (A \cup B)) = F$ or $(A \Rightarrow (A \cup B)) =\bot$. The first case is impossible because $A1$ is a classical tautology.
Consider the second case. $(A \Rightarrow (A \cup B)) =\bot$ in two cases.

c1 $A =\bot$ and $(A \cup B) = F$, i.e. $(\bot \cup B) = F$, what is impossible.

c2 $A = T$ and $(A \cup B) =\bot$, i.e. $(T \cup B) =\bot$, what is impossible.

c3 $A =\bot$ and $(A \cup B) =\bot$, i.e. $(\bot \cup B) =\bot$. This is possible for $B =\bot$ or $B = F$, i.e when $A =\bot, B =\bot$ or $A =\bot, B = F$.

From the above Observation we get second solution.

Solution 2: S is not sound under **K** semantics. Axiom A1 is not **K** semantics tautology. There are exactly two truth assignments v, such that $v \not\models A1$. One is, as defined in Solution 1: $A =\perp, B =\perp$. The second is $A =\perp, B = F$.

QUESTION 2

1. Write a formal proof A_1, A_2, A_3 in S from the QUESTION 1 with 2 applications of the rule (r) that starts with axiom A1, i.e such that $A_1 = A1$.

Solution: The formal proof A_1, A_2, A_3 is as follows.
$A_1 = (A \Rightarrow (A \cup B))$
Axiom
$A_2 = (A \Rightarrow (A \Rightarrow (A \cup B)))$
Rule (r) application for $A = A$ and $B = (A \cup B)$
$A_3 = (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B))))$
Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \cup B))$.

2. Use results from QUESTION 1 to determine whether $\models_{\mathbf{K}} A_3$.

Solution 1: We use the two v from QUESTION 3 to evaluate the logical value of A_3. Namely we evaluate: $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp\Rightarrow (\perp\Rightarrow (\perp\Rightarrow (\perp \cup \perp)))) =\perp$, or $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp\Rightarrow (\perp\Rightarrow (\perp\Rightarrow (\perp \cup F)))) =\perp$. Both cases prove that $\not\models_{\mathbf{K}} A_3$.

Solution 2: We know that S is not sound, because there is v for which $A1 = A_1 =\perp$, as evaluated in Question 3. We prove that the rule (r) preserves the logical value $\perp$ under any v such that $A1 =\perp$. as follows.
Let the premiss $(A \Rightarrow B) =\perp$, the logical value of the conclusion is hence $(A \Rightarrow\perp) =\perp$ for $A =\perp, T$ and $(A \Rightarrow\perp) = T$ for $A = F$.
The case $A = F$ evaluates the premiss $(A \Rightarrow B) = (F \Rightarrow B) = T$ for all B, what contradicts the assumption that $(A \Rightarrow B) =\perp$. We must hence have $A =\perp$. But all possible v for which $A1 =\perp$ are such that $A =\perp$, what end the proof.
It means that any A such that A has proof that starts with axiom $A1$ and then multiple applications of the rule (r) is evaluated to $\perp$ under all v, such that $v^*(A1) =\perp$. Hence, in particular, $\not\models_{\mathbf{K}} A_3$.

3. Write a formal proof A_1, A_2 in S from the QUESTION 3 with 1 application of the rule (r) that starts with axiom A2, i.e such that $A_1 = A2$.

Solution: The formal proof A_1, A_2 is as follows.
$A_2 = (A \Rightarrow (B \Rightarrow A))$
Axiom
$A_2 = (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$
Rule (r) application for $A = A$ and $B = (B \Rightarrow A)$.

4. Use results from QUESTION 1 to determine whether $\models A_2$.

Solution: System S is sound under classical semantics, hence by the soundness theorem we get that

$$\models (A \Rightarrow (A \Rightarrow (B \Rightarrow A))),$$

as it has a proof in S.

QUESTION 3 Prove, by constructing a formal proof in S from the QUESTION 3 that

$$\vdash_S (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))).$$

Solution: $A_2 = (A \Rightarrow (A \Rightarrow A))$

Axiom for $B = A$

$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow A)$.

$(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \Rightarrow A))$.