CSE451 EXERCISE 7 SOLUTIONS

Problem 1

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \quad \mathcal{E} = \mathcal{F} \quad AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \quad (r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}).$$

1. Prove that S is *sound* under classical semantics.

2. Prove that S is *not sound* under **K** semantics.

3. Write a formal proof in S with 2 applications of the rule (r).

Solution of 1.

Definition: System S is sound if and only if

(i) Axioms are tautologies and

(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

We verify the conditions (i), (ii) of the definition as follows.

(i) Both axioms of S are basic classical tautologies.

(ii) Consider the rule of inference of S.

$$(r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}).$$

Assume that its premise (the only premise) is True, i.e. let v be any truth assignment, such that $v^*(A \Rightarrow B) = T$. We evaluate logical value of the conclusion under the truth assignment v as follows.

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$.

Solution of 2. System S is *not sound* under **K** semantics because axiom $(A \Rightarrow A)$ is not a **K** semantics tautology.

Solution of 3. There are many solutions. Here is one of them.

Required formal proof is a sequence A_1, A_2, A_3, where
$A_1 = (A \Rightarrow A)$
(Axiom)
$A_2 = (A \Rightarrow (A \Rightarrow A))$
Rule (r) application 1 for $A = A,\ B = A$.
$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$.

Problem 2

Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))),$$

where S is the proof system from Problem 1.

Solution: Required formal proof is a sequence $A_1, A_2,$ where
$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$
Axiom
$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$
Rule (r) application for $A = A, B = (\neg A \Rightarrow B)$.

Observe that we needed only one application of the rule (r). One more application of the rule (r) to A_2 gives another solution to Problem 1, namely a proof A_1, A_2, A_3 for A_1, A_2 defined above and
$A_3 = ((A \Rightarrow (\neg A \Rightarrow B)) \Rightarrow (\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$
Rule (r) application for $A = (\neg A \Rightarrow B)$ and $B = (A \Rightarrow (\neg A \Rightarrow B))$.

Problem 3

Given a proof system:

$$S = (\mathcal{L}_{\{\cup, \Rightarrow\}}, \ \ \mathcal{E} = \mathcal{F} \ \ AX = \{A1, A2\}, \ \ \mathcal{R} = \{(r)\} \ ),$$

where

$$A1 = (A \Rightarrow (A \cup B)), \quad A2 = (A \Rightarrow (B \Rightarrow A))$$

and

$$(r) \ \frac{(A \Rightarrow B)}{(A \Rightarrow (A \Rightarrow B))}$$

Prove that S is *sound* under classical semantics.

Solution: Axioms of S are basic classical tautologies. The proof of soundness of the rule of inference is the following.

Assume $(A \Rightarrow B) = T$. Hence the logical value of conclusion is $(A \Rightarrow (A \Rightarrow B)) = (A \Rightarrow T) = T$ for all A.

Problem 4

Determine whether S from the Problem 3 is *sound* or *not sound* under **K** semantics.

Solution 1: S is not sound under **K** semantics. Let's take truth assignment such that $A = \perp, B = \perp$. The logical value of axiom A1 is as follows.
$(A \Rightarrow (A \cup B)) = (\perp \Rightarrow (\perp \cup \perp)) = \perp$ and $\not\models_{\mathbf{K}} (A \Rightarrow (A \cup B))$.

Observe that the v such that $A = \perp, B = \perp$ is not the only v that makes $A1 \neq T$, i.e. proves that $\not\models_{\mathbf{K}} A1$.
$(A \Rightarrow (A \cup B)) \neq T$ if and only if $(A \Rightarrow (A \cup B)) = F$ or $(A \Rightarrow (A \cup B)) = \perp$. The first case is impossible because $A1$ is a classical tautology.
Consider the second case. $(A \Rightarrow (A \cup B)) = \perp$ in two cases.

c1 $A = \perp$ and $(A \cup B) = F$, i.e. $(\perp \cup B) = F$, what is impossible.

c2 $A = T$ and $(A \cup B) = \perp$, i.e. $(T \cup B) = \perp$, what is impossible.

c3 $A = \perp$ and $(A \cup B) = \perp$, i.e. $(\perp \cup B) = \perp$. This is possible for $B = \perp$ or $B = F$, i.e when $A = \perp, B = \perp$ or $A = \perp, B = F$.

From the above Observation we get second solution.

Solution 2: S is not sound under **K** semantics. Axiom A1 is not **K** semantics tautology. There are exactly two truth assignments v, such that $v \not\models A1$. One is, as defined in Solution 1: $A = \perp, B = \perp$. The second is $A = \perp, B = F$.

Problem 5

Write a formal proof A_1, A_2, A_3 in S from the Problem 3 with 2 applications of the rule (r) that starts with axiom A1, i.e such that $A_1 = A1$.

Solution: The formal proof A_1, A_2, A_3 is as follows.
$A_1 = (A \Rightarrow (A \cup B))$
Axiom
$A_2 = (A \Rightarrow (A \Rightarrow (A \cup B)))$
Rule (r) application for $A = A$ and $B = (A \cup B)$
$A_3 = (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B))))$
Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \cup B))$.

Problem 6

Use results from Problem 4 to determine whether $\models_{\mathbf{K}} A_3$.

Solution 1: We use the two v from QUESTION 3 to evaluate the logical value of A_3. Namely we evaluate: $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp \Rightarrow (\perp \Rightarrow (\perp \Rightarrow (\perp \cup \perp)))) = \perp$, or $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp \Rightarrow (\perp \Rightarrow (\perp \Rightarrow (\perp \cup F)))) = \perp$. Both cases prove that $\not\models_{\mathbf{K}} A_3$.

Solution 2: We know that S is not sound, because there is v for which $A1 = A_1 = \perp$, as evaluated in Problem 4. We prove that the rule (r) preserves the logical value $\perp$ under any v such that $A1 = \perp$. as follows.
Let the premiss $(A \Rightarrow B) = \perp$, the logical value of the conclusion is hence $(A \Rightarrow \perp) = \perp$ for $A = \perp, T$ and $(A \Rightarrow \perp) = T$ for $A = F$.
The case $A = F$ evaluates the premiss $(A \Rightarrow B) = (F \Rightarrow B) = T$ for all B, what contradicts the assumption that $(A \Rightarrow B) = \perp$. We must hence have $A = \perp$. But all possible v for which $A1 = \perp$ are such that $A = \perp$, what end the proof.
It means that any A such that A has proof that starts with axiom $A1$ and then multiple applications of the rule (r) is evaluated to $\perp$ under all v, such that $v^*(A1) = \perp$. Hence, in particular, $\not\models_{\mathbf{K}} A_3$.

Problem 7

Write a formal proof A_1, A_2 in S from the Problem 3 with 1 application of the rule (r) that starts with axiom A2, i.e such that $A_1 = A2$.

Solution: The formal proof A_1, A_2 is as follows.
$A_2 = (A \Rightarrow (B \Rightarrow A))$
Axiom
$A_2 = (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$
Rule (r) application for $A = A$ and $B = (B \Rightarrow A)$.

Problem 8

Use results from Problem 3 to determine whether $\models A_2$.

Solution: System S is sound under classical semantics, hence by the Soundness Theorem we get that $\models (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$, as it has a proof in S.

Problem 9

Prove, by constructing a formal proof in S from the Problem 3 that

$$\vdash_S (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))).$$

Solution: $A_2 = (A \Rightarrow (A \Rightarrow A))$
Axiom for $B = A$
$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application for $A = A$ and $B = (A \Rightarrow A)$.
$(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \Rightarrow A))$.