cse547 Midterm 1 SOLUTIONS Fall 2023
(60pts + 10extra)

QUESTION 1 (10pts)

Given a Recursive Formula **RF**:

$$f(1) = \alpha; \quad f(2n + j) = 2f(n) + \beta_j, \quad \text{where} \quad \beta_j \in Z, \quad j = 0, 1, \quad n \geq 1$$

Given $k = (b_m, b_{m-1}, ...b_1, b_0)_2$. We want to evaluate: $f(k) = f((b_m, b_{m-1}, ...b_1, b_0)_2)$

1. (5pts) **Prove** that dealing with "normal" **binary representation** we do not need to consider cases of $\mathbf{k} \in$ **odd** or $\mathbf{k} \in$ **even** when using the recursive formula **RF**. Consider cases: $k = 2n = (b_m, b_{m-1}, ...b_1, b_0)_2$ and $k = 2n + 1 = (b_m, b_{m-1}, ...b_1, b_0)_2$ and evaluate n in both cases.

Solution

The binary representation of **k=2n** is given as:

$$2n = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

We get $b_m = 1$ and $b_0 = 0$

Hence,

$$n = 2^{m-1} b_m + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

The binary representation of **k=2n +1** is given as:

$$2n + 1 = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n + 1 = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

$$b_0 = 1, b_m = 1$$

Hence,

$$n = 2^{m-1} b_m + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

2. (5pts)

We **define** a **relaxed binary** representation as $\mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \beta_{\mathbf{b_0}} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_\mathbf{2}$, where β_{b_k} are defined as follows

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

The closed formula **CF** for **RF** is $f((b_m, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_2$.

Evaluate $f(19)$ for original Josephus, i.e. for $\alpha = 1$, $\beta_0 = -1$, $\beta_1 = 1$

Solution

$19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0 = (10011)_2$

$f(19) = f((10011)_2) = (1 - 1 - 111)_2 = 7$

QUESTION 2 (10pts)

Any recurrence of the type $a_nT_n = b_nT_{n-1} + c_n$ for $n \geq 1$ and T_0 given by an initial condition has a **CF** formula

$$T_n = \frac{1}{a_n s_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$$

where the summation factor s_k is given by $s_n = s_1 \frac{a_1 a_2 .....a_{n-1}}{b_2 b_3 b_4 ....b_n}$, and s_1 is a constant

Use the **CF** formula and the summation factor to solve the recurrence

$$T_0 = 5$$

$$2T_n = nT_{n-1} + 3n! \quad \text{for } n > 0$$

Solution

We use $s_n = s_1 \frac{a_1 a_2 .....a_{n-1}}{b_2 b_3 b_4 ....b_n}$ for $a_n = 2$ and b_n = n and get $s_n = s_1 \frac{a_{n-1}...a_1}{b_n...b_2} = \frac{2{\cdot}2 \ ... \ 2{\cdot}2}{n(n-1)...2.1} = \frac{2^{n-1}}{n!}$ and $s_1 = 1$

We have that $T_0 = 5$, $c_n = 3$ n!, b_n = n, and we can substitute in the closed formula for T_n

$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k) = \frac{n!}{2^n}(5 + 3 \sum_{k=1}^{n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3 \sum_{k=1}^{n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3 \sum_{1 \leq k \leq n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3 \sum_{0 \leq k-1 \leq n-1} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3 \sum_{r=0}^{n-1} 2^r)$, where we set r = k - 1

We know that that $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$, for $x \neq 1$, so in our case

$T_n = \frac{n!}{2^n}(5 + 3 \sum_{r=0}^{n-1} 2^r)$

$= \frac{n!}{2^n}(5 + 3\frac{2^{(n-1)+1}-1}{2-1})$

$= \frac{n!}{2^n}(2 + 3 \cdot 2^n)$

= n! $(2^{(1-n)} + 3)$

QUESTION 3 (15pts)

Use the Perturbation Method to evaluate a closed formula for the following (assuming that $n \geq 0$).

$$S_n = \sum_{k=0}^{n} (-1)^{n-k}$$

Hint Split the S_n first on the first term, then on the last term, and combine the results

Solution

Split off the first term

$$
\begin{aligned}
S_{n+1} &= \sum_{k=0}^{n+1}(-1)^{n+1-k} \\
&= (-1)^{n+1-0} + \sum_{k=1}^{n+1}(-1)^{n+1-k} \\
&= (-1)^{n+1} + \sum_{k+1=1}^{n+1}(-1)^{n+1-(k+1)} \\
&= (-1)^{n+1} + \sum_{k=0}^{n}(-1)^{n-k} \\
&= (-1)^{n+1} + S_n
\end{aligned}
$$

Split off the last term

$$
\begin{aligned}
S_{n+1} &= \sum_{k=0}^{n+1}(-1)^{n+1-k} \\
&= \sum_{k=0}^{n}(-1)^{n+1-k} + (-1)^{n+1-(n+1)} \\
&= \sum_{k=0}^{n}(-1)^{n+1-k} + 1 = \sum_{k=0}^{n}(-1)^{n-k}(-1)^{1} + 1 \\
&= -\sum_{k=0}^{n}(-1)^{n-k} + 1 \\
&= -S_n + 1
\end{aligned}
$$

From the above two equations we evaluate

$$
\begin{aligned}
(-1)^{n+1} + S_n &= -S_n + 1 \\
S_n &= \frac{1}{2}(1 - (-1)^{n+1}) \\
S_n &= \frac{1}{2}(1 + (-1)^{n})
\end{aligned}
$$

QUESTION 4 (25pts) + (10extra)

Use **repertoire method** to evaluate a closed form formula **CF**

$R_n = \alpha A(n) + \beta B(n) + \gamma C(n) + \delta D(n)$

for the **general form** of the recurrence **RF**

$R_0 = \alpha$

$R_n = R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2)$

Use functions: $R(n) = 1$, $R(n) = (-1)^n$, $R(n) = (-1)^n n$, $R(n) = (-1)^n n^2$, for all $n \in N$

1. (15pts) Find the equations **E1 - E3** for $A(n)$, $B(n)$, $C(n)$ of the closed form formula **CF** and evaluate all their components.

The fourth equation **E4** for $D(n)$ is: $D(n) = \frac{(-1)^n n^2 + 2C(n) - B(n)}{2}$

The fourth equation **E4** evaluated on $D(n)$ is $D(n) = \frac{n(n+1)(-1)^n}{2}$

Solution

We take the repertoire function $R_n = 1$ for all $n \in N$ and plug it in to RF

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + (-1)^n(\beta + \gamma n + \delta n^2) \\ 0 &= (-1)^n(\beta + \gamma n + \delta n^2) \\ 0 &= \beta + \gamma n + \delta n^2 \end{aligned}$$

This holds for all $n \in N$ if and only if $\alpha = 1, \beta = 0, \gamma = 0, \delta = 0$.

Our **first equation E1** is :

$$A(n) = 1$$

We take the repertoire function $R_n = (-1)^n$ for all $n \in N$ and plug it in to RF

$$\begin{aligned} (-1)^0 &= \alpha \\ 1 &= \alpha \end{aligned}$$

$$\begin{aligned} (-1)^n &= (-1)^{n-1} + (-1)^n(\beta + \gamma n + \delta n^2) \\ 2(-1)^n &= (-1)^n(\beta + \gamma n + \delta n^2) \\ 2 &= \beta + \gamma n + \delta n^2 \\ 0 &= (\beta - 2) + \gamma n + \delta n^2 \end{aligned}$$

This holds for all $n \in N$ if and only if $\alpha = 1, \beta = 2, \gamma = 0, \delta = 0$.

We take the repertoire function $R_n = (-1)^n$ for all $n \in N$ and found $\alpha = 1, \beta = 2, \gamma = 0, \delta = 0$ and plug it in to CF.

$$(-1)^n = A(n) + 2B(n)$$

The **second equation E2** is

$$B(n) = \frac{(-1)^n - A(n)}{2}$$

Use **E1** and solve it on B(n)

$$\begin{aligned} 2B(n) &= (-1)^n - 1 \\ B(n) &= \frac{(-1)^n - 1}{2} \end{aligned}$$

The **second equation E2** solved on B(n) is

$$B(n) = \frac{(-1)^n - 1}{2}$$

Third equation **E3**

We take the repertoire function $R_n = (-1)^n n$ for all $n \in N$ and plug it in to RF

$$\begin{aligned}(-1)^0 \cdot 0 &= \alpha \\ 0 &= \alpha\end{aligned}$$

$$\begin{aligned}(-1)^n n &= (-1)^{n-1}(n-1) + (-1)^n(\beta + \gamma n + \delta n^2) \\ (-1)^n n &= (-1)^n(-1)(n-1) + (-1)^n(\beta + \gamma n + \delta n^2) \\ (-1)^n n &= (-1)^n(1-n) + (-1)^n(\beta + \gamma n + \delta n^2) \\ (n+n-1)(-1)^n &= (-1)^n(\beta + \gamma n + \delta n^2) \\ (2n-1) &= (\beta + \gamma n + \delta n^2) \\ 0 &= (\beta + 1) + (\gamma - 2)n + \delta n^2\end{aligned}$$

This holds for all $n \in N$ if and only if $\alpha = 0, \beta = -1, \gamma = 2, \delta = 0$.

We take the repertoire function $R_n = (-1)^n n$ for all $n \in N$ and plug it in to CF with $\alpha = 0, \beta = -1, \gamma = 2, \delta = 0$. We evaluate

$$\begin{aligned}(-1)^n n &= -B(n) + 2C(n) \\ 2C(n) &= (-1)^n n + B(n)\end{aligned}$$

The **third equation E3** is

$$C(n) = \frac{(-1)^n n + B(n)}{2}$$

We evaluate C(n) using **E2**

$$\begin{aligned}2C(n) &= (-1)^n n + \frac{(-1)^n - 1}{2} \\ 4C(n) &= 2n(-1)^n + (-1)^n - 1\end{aligned}$$

The **third equation E3** solved on $C(n)$ is

$$C(n) = \frac{(2n+1)(-1)^n - 1}{4}$$

Thus putting the derived values for A(n), B(n), C(n) and D(n) in the **CF** for R_n, we get,

$$R_n = \alpha + \beta\frac{(-1)^n - 1}{2} + \gamma\frac{((-1)^n(2n+1) - 1)}{4} + \delta\frac{((-1)^n(n+n^2))}{2}$$

2. (10pts) Use the closed formula **CF** for the **general form** of the recurrence **RF** to evaluate

$$S_n = \sum_{k=0}^{n} (-1)^k k^2$$

Solution

The recurrence form of the summation $S_n = \sum_{k=0}^{n} (-1)^k k^2$ is

R

$S_0 = 0$

$S_n = S_{n-1} + (-1)^n n^2$

Observe that it s a particular form of of the recurrence

RF

$R_0 = \alpha$

$R_n = R_{n-1} + (-1)^n(\beta + \gamma n + \delta n^2)$

and its closed formula **CF** $R_n = \alpha A(n) + \beta B(n) + \gamma C(n) + \delta D(n)$ for $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 1$

Hence the closed formula for our sum is

$$S_n = D(n)$$
$$S_n = \frac{n(n+1)(-1)^n}{2}$$

3. (10pts) Extra Credit

Prove and evaluate the **Equation 4** for $D(n)$

Solution

We take the repertoire function $R_n = (-1)^n n^2$ and plug it in to RF

$$(-1)^0 0^2 = \alpha$$
$$0 = \alpha$$

$$(-1)^n n^2 = (-1)^{n-1}(n-1)^2 + (-1)^n(\beta + \gamma n + \delta n^2)$$
$$(-1)^n n^2 = (-1)^{-1}(-1)^n(n-1)^2 + (-1)^n(\beta + \gamma n + \delta n^2)$$
$$(n^2 + (n-1)^2) = (-1)^n(\beta + \gamma n + \delta n^2)$$
$$(n^2 + n^2 - 2n + 1)(-1)^n = (-1)^n(\beta + \gamma n + \delta n^2)$$
$$(2n^2 - 2n + 1)(-1)^n = (-1)^n(\beta + \gamma n + \delta n^2)$$
$$2n^2 - 2n + 1 = \beta + \gamma n + \delta n^2$$
$$0 = (\beta - 1) + (\gamma + 2)n + (\delta - 2)n^2$$

This will give $\alpha = 0, \beta = 1, \gamma = -2, \delta = 2$. So our **fourth equation E4** is :

$$(-1)^n n^2 = B(n) - 2C(n) + 2D(n)$$
$$D(n) = \frac{(-1)^n n^2 + 2C(n) - B(n)}{2}$$

We evaluate the fourth equation on $D(n)$

$$(-1)^n n^2 = B(n) - 2C(n) + 2D(n)$$
$$2n^2(-1)^n = (-1)^n - 1 - ((2n+1)(-1)^n - 1) + 4D(n)$$
$$4D(n) = 2n^2(-1)^n - (-1)^n + 1 + (2n+1)(-1)^n - 1$$
$$4D(n) = 2n^2(-1)^n - (-1)^n + (2n+1)(-1)^n$$
$$4D(n) = (2n^2 + 2n - 1 + 1)(-1)^n$$
$$4D(n) = (2n^2 + 2n)(-1)^n$$

The **fourth equation E4** evaluated on $D(n)$ is

$$D(n) = \frac{n(n+1)(-1)^n}{2}$$