CSE547 SOME EXERCISES on SETS SOLUTIONS

FINITE and INFINITE SETS

Definition 1

A set A is FINITE iff there is a natural number $n \in N$ and there is a $1-1$ function f that maps the set $\{1, 2, ...n\}$ onto A.

Definition 2

A set A is INFINITE iff it is NOT FINITE.

QUESTION 1

Use the above definitions to prove the following

FACT 1 *A set A is INFINITE if and only if it contains a countably infinite subset, i.e. one can define a* $1-1$ *sequence* $\{a_n\}_{n \in N}$ *of some elements of* A

SOLUTION

S1. Proof of Implication

If A is infinite, then we can define a 1-1 sequence of elements of A

Let A be infinite. We define a sequence $a_1, \ \dots, \ a_n, \ \dots\dots$ as follows.

1. Observe that $A \neq \emptyset$, because if $A = \emptyset$, A would be finite. Contradiction. So there is an element of $a \in A$. We define

$$a_1 \ = \ a$$

2. Consider a set $A - \{a_a\} = A_1$. $A_1 \neq \emptyset$ because if $A = \emptyset$, then $A - \{a_1\} = \emptyset$ and A is Finite. Contradiction. So there is an element $a_2 \in A - \{a_1\}$ and $a_1 \neq a_2$.
We defined

$$a_1, \ a_2$$

$$a_{1,2}, \dots, a_n \text{for } a_1 \neq a_2 \neq \dots \neq a_n$$

Assume that we defined a set $A_n = A - \{a_1, \dots, a_n\}$.
The set $A_n \neq \emptyset$ because if $A - \{a_1, \dots, a_n\} = \emptyset$,then A is finite.Contradiction.
So there is an element

$$a_{n+1} \in A - \{a_1, \dots, a_n\}$$

and $a_{n+1} \neq a_n \neq \dots \neq a_1$
By mathematical induction, we have defined a 1-1 sequence

$$a_1, a_2, \dots, a_n, \dots$$

elements of A.
2. Implication $\leftarrow$
If A contain a 1-1 sequence, then A is infinite.
Assume A is not infinite; i.e A is finite. Every subset of finite set is finite, so we can't have a 1-1 infinite sequence of elements of A. Contradiction.

QUESTION 2 Use the above definitions and FACT 1 from QUESTION 1the following characterization of infinite sets.

Dedekind Theorem A set A is INFINITE iff there is a set proper subset B of the set A such that $|A| = |B|$.

SOLUTION Part1. If A is infinite, then there is $B \subsetneq A$ and

$$f : A \xrightarrow[onto]{1-1} B$$

A is infinite, by Q1, we have a 1-1 sequence

$$a_1, a_2, \dots, a_n, \dots$$

of elements A.
We take $B = A - \{a_1\}$, $B \subsetneq A$ and we define a function

$$f : A \xrightarrow[onto]{1-1} B$$

as follows

$$f(a_1) = a_2$$
$$f(a_2) = a_3$$
$$\vdots$$
$$f(a_n) = a_{n+1}$$
$$f(a) = a, \text{for all other } a \in A$$

obviously, f is 1-1,onto
Observe: we have other choises of B!.
Part 2. Assume that we have $B \subsetneq A$ are

$$f : A \xrightarrow[onto]{1-1} B$$

We use Q1 to show that A is infinite; i.e we construct an 1-1 sequence $a_1 \dots a_n$ of elements of A_n as follows.
$B \subsetneq A$,so $A - B \neq \emptyset$ and we have $b \in A - B$. This is our first element of the sequence.
Observe: $f : A \xrightarrow[onto]{1-1} B$, so $f(b) \in B$ and $b \in A - B$, hence $f(b) \neq b$ and $f(b)$ is our second element of the sequence.
We have now,
$b, f(b) \qquad f(b) \neq b, b \in A - B, f(b) \in B$
Take new,
$ff(b)$.As f is 1-1 and $f(b) \neq b$, we get $ff(b) \neq f(b) \neq b, ff(b) \in B$ and the sequence $b, f(b), ff(b)$ is 1-1.

We create $ff(b) = f^2(b)$
We continue the construction by mathematical induction.
Assume that we have constructed a 1-1 sequence

$$b, f(b), f^(b), f^3(b), \ldots, f^n(b)$$

Observe that $ff^n(b) = f^{n+1}(b) \neq f^n(b)$ as f is 1-1.
By mathematical induction, we have that $\{f^n(b)\}_{n \in N}$ is a 1-1 sequence of elements of A and hence A is infinite.

QUESTION 3 Use technique from DEDEKIND THEOREM to prove the following

Theorem For any infinite set A and its finite subset B, $|A| = |A - B|$.

SOLUTION A is infinite, then by Q1 there is a 1-1 sequence:

$$a_1, a_2, \ldots, a_n, \ldots$$

of elements of A.
Let $|B| = k$. We choose k 1-1 sequences $\{c_n^k\}_{n \in N}$ of the sequence $\{a_n\}_{n \in N}$, such that $c_n^j \neq c_n^i$ for all $j \neq i, 1 \leq i, j \leq k$ and all $n \in N$.
Let $B = \{b_1, \ldots, b_k\}$.We construct a function $f : A \xrightarrow[onto]{1-1} A - \{b_1, \ldots, b_k\}$ as follows

$$f(b_1) = c_1^1, \qquad f(c_1^1) = c_2^1, \ldots, f(c_n^1) = c_{n+1}^1$$
$$f(b_2) = c_1^2, \qquad f(c_1^2) = c_2^2, \ldots, f(c_n^2) = c_{n+1}^2$$
$$\vdots$$
$$f(b_k) = c_1^k, \qquad f(c_1^k) = c_2^k, \ldots, f(c_n^k) = c_{n+1}^k$$
$$f(a) = a \text{ all } a \in A - B$$

As all sequences $\{C_n^m\}_{n \in N, m=1,\ldots,k}$ are 1-1, and different, the function f is 1-1 and obviously ONTO $A - B$.

QUESTION 4 Use DEDEKIND THEOREM to prove that the set N of natural numbers is infinite.

SOLUTION We use Dedekind theorem i.e we must define $f : N \xrightarrow[onto]{1-1} B \subsetneq N$. There are many such function for example $f(n) = n + 1$.$f : N \xrightarrow[onto]{1-1} N - \{0\}$
One can also use Q1 and define any 1-1 sequences in N.

QUESTION 5 Use DEDEKIND THEOREM to prove that the set R of real numbers is infinite.

SOLUTION We use Dedekind theorem

$$f(x) = 2^x \qquad x \in R$$
$$f : R \xrightarrow[onto]{1-1} R^+$$

One can also use Q1 and define any 1-1 sequences in R.

QUESTION 6 Use technique from DEDEKIND THEOREM to prove that the interval $[a, b], a < b$ of real numbers is infinite and that $|[a, b]| = |(a, b)|$.

SOLUTION1 Use construction in the proof of Q3.
$f : [a, b] \xrightarrow[onto]{1-1} [a, b] - \{a, b\} = (a, b)$
This is the soution I had in mine!

SOLUTION2 Use Q3 $(a,b) = [a,b] - B, B$:finite

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers $\mathcal{M}, \mathcal{N}, \mathcal{K}$ the following formulas hold:

$$1. \mathcal{N} \leq \mathcal{N}$$

$$2. If \;\; \mathcal{N} \leq \mathcal{M} \;\; and \;\; \mathcal{M} \leq \mathcal{K}, \;\; then \;\; \mathcal{N} \leq \mathcal{K}.$$

SOLUTION 1. $\mathcal{N} \leq \mathcal{N}$ means that for any set A, $|A| \leq |A|$

$f(a) = a$ establishes $f : A \xrightarrow{1-1} A$

2. $\mathcal{N} \leq \mathcal{M}$ *and* $\mathcal{M} \leq \mathcal{K}$, *then* $\mathcal{N} \leq \mathcal{K}$.

We have $|A| = \mathcal{N}$,$|b| = \mathcal{M}$,$|C| = \mathcal{K}$ and $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} C$,then we have to construct $h : A \xrightarrow{1-1} C$.

h is a composition of f and g. i.e $h(a) = g(f(a))$, all $a \in A$

QUESTION 8 Prove, for any sets A, B, C the following holds.

Fact 2

$$If \;\; C \subseteq B \subseteq A \; and \; |A| = |C|, \;\; then \;\; |A| = |B| = |C|.$$

To prove $|A| = |B|$ you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3

SOLUTION 1. A, B, C are finite and $|A| = |C|$, and $C \subseteq B \subseteq A$,so $A = B = C$, and have $|A| = |B| = |C|$

2.A, B, C are infinite sets, we have $|A| = |C|$ i.e we have $f : A \xrightarrow[onto]{1-1} C$

We want to construct a function

$$g : A \xrightarrow[onto]{1-1} B, \text{ where } A \subseteq B \subseteq C$$

Take $A - B$. We assume that $A - B \neq \emptyset$, if not, $A = B$, and $|A| = |C|$ given $|A| = |B| = |C|$.

We consider case $C \subset B \subset A$. Take any $a \in (A - B)$, as $f : A \xrightarrow[onto]{1-1} C$, $f(a) \in C$,f is 1-1 so $ff(a) \neq f(a)$ $\ldots$ in general $f^n(a) \neq f^{n+1}(a)$ and we have a sequence for any $a \in A - B$

$f(a), f^2(a), \ldots, f^n(a) \ldots$ of elements of C.

We construct a function $g : A \xrightarrow[onto]{1-1} B$

$$g(a) = f(a)$$

$$g(f(a)) = f^2(a)$$

$$g(f^2(a)) = f^3(a))$$

$$\vdots$$

$$g(f^n(a)) = f^{n+1}(a)$$

$$g(x) = x \qquad \text{for all other } x \in A$$

QUESTION 9 Prove the following

Berstein Theorem (1898) For any cardinal numbers $\mathcal{M}, \mathcal{N}$

$$\mathcal{N} \leq \mathcal{M} \;\; and \;\; \mathcal{M} \leq \mathcal{N} \;\; then \mathcal{N} = \mathcal{M}.$$

1. Prove first the case when the sets A, B are disjoint.

2. Generalize the construction for 1. to the not-disjoint case.