Discrete Mathematics

Chapter 1, Problem 18,19

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Problem

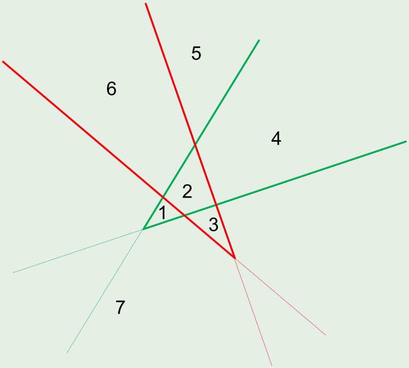
Problem 18

Show that the following set of n bent lines defines Z_n regions, where Z_n is defined in (1.7): The jth bent line, for $i \leqslant j \leqslant n$, has its zig at $(n^{2j}, 0)$ and goes up through the points $(n^{2j} - n^j, 1)$ and $(n^{2j} - n^j - n^{-n}, 1)$.

Analysis

How can we derive $Z_n = 2n^2 - n + 1$?

Example 1



Analysis

A bent line is like two straight lines except that regions merge when the "two" lines don't extend past their intersection point. To obtain Z_n, one requirement is that: each ray should intersect with all other rays. So the situation is similar to $2n$ lines arrangement, but we lose only two regions per line.

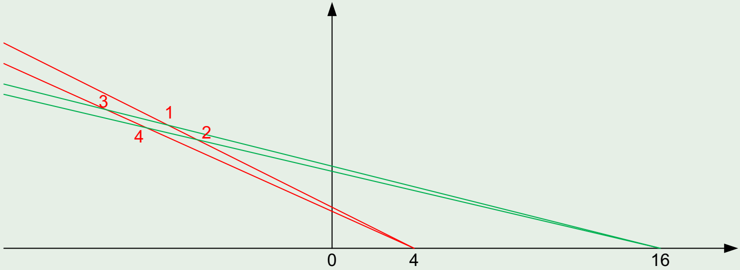
Close formula of Z_n

$$\begin{aligned} Z_n &= L_{2n} - 2n = 2n(2n+1)/2 + 1 - 2n \\ &= 2n^2 - n + 1 \end{aligned} \tag{1}$$

Solution

This is the situation when $n = 2$.

Example 2



Solution

We should prove

Every ray intersects with other rays at distinct points.

The slopes of the jth two rays are $-1/n^j$ and $-1/[n^j + n^{-n}]$ respectively. Similarly, the slopes of the kth two rays are $-1/n^k$ and $-1/[n^k + n^{-n}]$ respectively.

Solution

The jth bent line can be expressed as follows $(1 \leqslant j \leqslant n)$:

$$\begin{cases} y = -\dfrac{1}{n^j}(x - n^{2j}) \\ y = -\dfrac{1}{n^j + n^{-n}}(x - n^{2j}) \end{cases} \tag{2}$$

The kth bent line can be expressed in similar way $(1 \leqslant k \leqslant n)$:

$$\begin{cases} y = -\dfrac{1}{n^k}(x - n^{2k}) \\ y = -\dfrac{1}{n^k + n^{-n}}(x - n^{2k}) \end{cases} \tag{3}$$

Solution

Suppose $k < j$, by solving equation 2 and 3, we can get the x coordinates of the intersection points, which are:

1. $-n^{j+k}$
2. $(n^{2j+k} + n^{2j-n} - n^{2k+j})/(n^k + n^{-n} - n^j)$
3. $(n^{2k+j} + n^{2k-n} - n^{2j+k})/(n^j + n^{-n} - n^k)$
4. $-n^{j+k} - n^{k-n} - n^{j-n}$

We can simplify the formula in the following way:

$$\begin{aligned} \frac{n^{2j+k} + n^{2j-n} - n^{2k+j}}{n^k + n^{-n} - n^j} &= n^{2j} + \frac{n^{3j} - n^{2k+j}}{n^k + n^{-n} - n^j} \\ &> n^{2j} + \frac{n^{3j} - n^{2k+j}}{n^k - n^j} \\ &= n^{2j} + \frac{n^j(n^j + n^k)(n^j - n^k)}{n^k - n^j} \\ &= -n^{j+k} \end{aligned} \tag{4}$$

Solution

$$\begin{aligned}
\frac{n^{2j+k}+n^{2j-n}-n^{2k+j}}{n^k+n^{-n}-n^j} &< \frac{n^{j+k}(n^j-n^k)}{n^k+n^{-n}-n^j} \\
&< \frac{n^{j+k}(n^{j-1}-n^{k-1})}{n^k+n^{-n}-n^j} \\
&< \frac{n^{2j+k-1}-n^{j+2k-1}-n^{j+k-n-1}}{n^k+n^{-n}-n^j} \\
&= -n^{j+k-1}
\end{aligned} \tag{5}$$

Solution

So, we have

$$-n^{j+k} \tag{6}$$

$$-n^{j+k} < \frac{n^{2j+k} + n^{2j-n} - n^{2k+j}}{n^k + n^{-n} - n^j} < -n^{j+k-1} \tag{7}$$

$$-n^{j+k+1} < \frac{n^{2k+j} + n^{2k-n} - n^{2j+k}}{n^j + n^{-n} - n^k} < -n^{j+k} \tag{8}$$

$$-n^{j+k} - n^{k-n} - n^{j-n} \tag{9}$$

Solution

Because four intersection points in Figure 2 belong to four distinct rays, they are distinct apparently. Further, we should consider whether there is a bent line i, such that i, j, k have a common intersection point.

Example 3

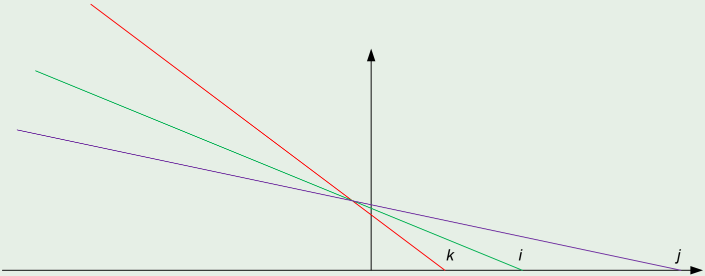
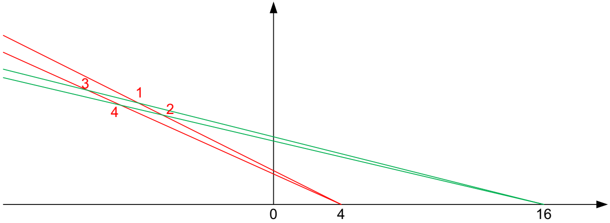


Figure: Is this situation possible?

Solution

First, we classify the intersection points into four groups:

1. upper-upper
2. upper-lower
3. lower-upper
4. lower-lower



Solution

Second, we choose the ray with largest x intercept as j. We use $v_{ij,\alpha}$ $(1 \leqslant \alpha \leqslant 4)$ to represent four intersection points of bent line i and j, and similarly $v_{kj,\alpha}$ stands for the intersection points of bent line k and j. α stands for the group. Now we should prove three claims:

1. $v_{ij,\alpha} \neq v_{kj,\alpha}$
2. $v_{ij,1} \neq v_{kj,3}$
3. $v_{ij,2} \neq v_{kj,4}$

Solution

From equations 6 and 9, we know that the x coordinates of the 1st class and the 4th class intersection points are $-n^{j+k}$ and $-n^{j+k} - n^{k-n} - n^{j-n}$, which are strictly monotonic when we fix j. That means the intersection points are also distinct when k are distinct. From equations 7 and 8, when we fix j, and let k be distinct values, the x coordinates obtained by the formulas are in the disjoint slots, so they are distinct. Therefore, $v_{ij,\alpha} \neq v_{kj,\alpha}$.

Solution

We first prove $v_{ij,1} \neq v_{kj,3}$. Because we know that x coordinate of $v_{ij,1}$ is $-n^{j+k}$, and x coordinate of $v_{kj,3}$ is between $-n^{j+k+1}$ and $< -n^{j+k}$, we can conclude that they cannot be the same.

Solution

We prove $v_{ij,2} \neq v_{kj,4}$ as follows:

- When $i < k$, suppose $v_{ij,2} = v_{kj,4}$, we can get $v_{ij,2} = v_{kj,4} = v_{ik,2}$, because we know that when $j \neq k$, $v_{ij,2} \neq v_{ik,2}$, we get a contradiction. Therefore, in this case, $v_{ij,2} \neq v_{kj,4}$
- When $i > k$, suppose $v_{ij,2} = v_{kj,4}$, we can get $v_{ij,2} = v_{kj,4} = v_{ki,3}$, because we know that $-n^{i+j} < x(v_{ij,2}) < -n^{i+j-1}$ and $-n^{k+i+1} < x(v_{ki,3}) < -n^{k+i}$ and $k \leqslant j - 2$, we get $x(v_{ij,2}) < x(v_{ki,3})$. Therefore, $v_{ij,2} \neq v_{kj,4}$.

Solution

In summary, because the x coordinates of all intersection points are less than n^{2k}, that means for each ray, it intersects with other rays. Further, we have proved that these rays intersect at distinct points. So this set of n bent lines defines Z_n regions.

Problem

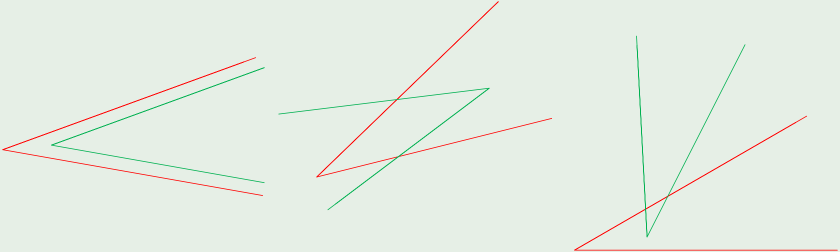
Problem 19

Is it possible to obtain Z_n regions with n bent lines when the angle at each zig is $30°$?

Analysis

We have a claim that: if one zig is located inside the bent line, in this situation, we cannot obtain Z_n regions.

Example 4



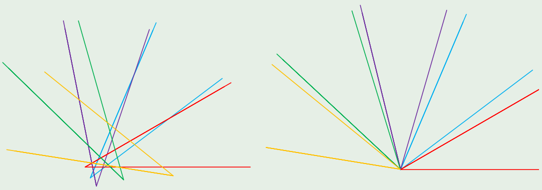
(a) Case 1 (b) Case 2 (c) Case 3

Figure: Zig is in the bent line

Solution

Base on the claim above, when add bent lines to the plane, we should avoid placing zig in any wedge.

Example 5



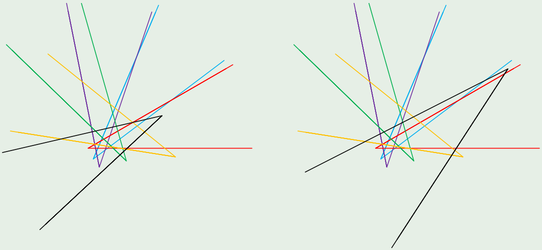
(a) (b)

Figure: After adding 5 bent lines

Solution

After adding 5 bent lines, we cannot place any additional bent line such that no zig lies in the region of other bent line. Therefore, if $n > 5$, it is impossible to obtain Z_n.

Example 6



(a) (b)

The end

Thank you!