CSE 547 — Problem 19, Chapter 2

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Outline

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The problem

Use a summation factor to solve the recurrence

$$\begin{aligned} T_0 &= 5 \\ 2T_n &= nT_{n-1} + 3\,n!, \qquad \text{for } n > 0 \end{aligned}$$

Looking at small cases. . .

Substituting in $2T_n = nT_{n-1} + 3\,n!$, we get:

- $T_0 = 5$
- $T_1 = \frac{1 \cdot 5 + 3 \cdot 1}{2} = 4$
- $T_2 = \frac{2 \cdot 4 + 3 \cdot 2 \cdot 1}{2} = 7$
- $T_3 = \frac{3 \cdot 7 + 3 \cdot 3 \cdot 2 \cdot 1}{2} = \frac{39}{2}$
- $T_4 = \frac{4 \cdot \frac{39}{2} + 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = 75$
- $T_5 = \frac{5 \cdot 75 + 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = \frac{735}{2}$

Looking at small cases...

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We cannot guess any pattern.

Which method can we use?

We can reduce the recurrence to a sum.

The general form is $a_n T_n = b_n T_{n-1} + c_n$
and comparing to our case $2 T_n = n T_{n-1} + 3\, n!$
we can see that

- $a_n = 2$
- $b_n = n$
- $c_n = 3\, n!$

Outline

How to reduce a recurrence to a sum

By multiplying by a summation factor s_n on both sides of

$$a_n T_n = b_n T_{n-1} + c_n$$

we get

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n$$

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If we are able to impose

$$s_n b_n = s_{n-1} a_{n-1}$$

then we can rewrite the recurrence as

$$S_n = S_{n-1} + s_n c_n$$

where $S_n = s_n a_n T_n$

Here is the recipe

(more details in the lecture's slides)

Expanding S_n, we get

$$S_n = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k$$

and then the closed formula for T_n is

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

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By unfolding $s_n = s_{n-1}a_{n-1}/b_n$, we obtain

$$s_n = \frac{a_{n-1}a_{n-1}\cdots a_1}{b_n b_{n-1}\cdots b_2}$$

Outline

In our case

Since $a_n = 2$ and $b_n = n$

$$s_n = \frac{a_{n-1}a_{n-1}\cdots a_1}{b_n b_{n-1}\cdots b_2} = \frac{\overbrace{2\cdot 2\cdot 2\cdots 2}^{n-1\ \text{times}}}{n\cdot(n-1)\cdots 2\cdot 1} = \frac{2^{n-1}}{n!}$$

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Remembering also that $T_0 = 5$ and $c_n = 3\,n!$,
we can substitute in the closed formula for T_n

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right) = \frac{n!}{2^n}\left(5 + 3\sum_{k=1}^{n} 2^{k-1}\right)$$

Let's simplify the sum...

$$
\begin{aligned}
T_n &= \frac{n!}{2^n}\left(5 + 3\sum_{k=1}^{n} 2^{k-1}\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{1\le k\le n} 2^{k-1}\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{0\le k-1\le n-1} 2^{k-1}\right) \\
&= \frac{n!}{2^n}\left(5 + 3\sum_{r=0}^{n-1} 2^{r}\right)
\end{aligned}
$$

where we set $r = k - 1$

And finally we get the solution! :-)

We have seen that

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}, \qquad \text{for } x \neq 1$$

so in our case

$$\begin{aligned} T_n &= \frac{n!}{2^n}\left(5 + 3\sum_{r=0}^{n-1} 2^r\right) \\ &= \frac{n!}{2^n}\left(5 + 3\,\frac{2^{(n-1)+1}-1}{2-1}\right) \\ &= \frac{n!}{2^n}(2 + 3\cdot 2^n) \\ &= n!\,(2^{1-n}+3) \end{aligned}$$

Outline

Let's double check the result

Substituting in $T_n = n!\,(2^{1-n} + 3)$, we get:

- $T_0 = 1 \cdot (2^{1-0} + 3) = 5$
- $T_1 = 1 \cdot (2^{1-1} + 3) = 4$
- $T_2 = 2 \cdot 1 \cdot (2^{1-2} + 3) = 7$
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which is exactly what we expected!

We showed that. . .

The solution of

$$\begin{aligned} T_0 &= 5 \\ 2\,T_n &= n\,T_{n-1} + 3\,n!, \qquad \text{for } n > 0 \end{aligned}$$

is

$$T_n = n!\,(2^{1-n} + 3)$$

Jumping ahead. . . :-)

Let me ask you this question:

When is

$$T_n = n!\,(2^{1-n} + 3)$$

an integer?

Later on in this class, we will learn how to show that T_n is integer iff n is 0 or a power of 2.

Questions ?

Junkil Choo, Giordano Fusco, Nancy Ju

Thanks!