CSE547

Chapter 2, Problem 23

Problem 23

Evaluate the sum $\sum_{k=1}^{n} \frac{(2k+1)}{(k(k+1))}$ in two ways :

a. Replace $\frac{1}{k(k+1)}$ by partial fractions

b. Sum by parts

Problem 23 (a)

$$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$$

$$\frac{2k+1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

$$\frac{2k+1}{k(k+1)} = \frac{A(k+1)+Bk}{k(k+1)}$$

$$\frac{2k+1}{k(k+1)} = \frac{(A+B)k+A}{k(k+1)}$$

Comparing both the sides…

$$A = 1; \; A + B = 2$$

$$\Rightarrow A = 1, B = 1$$

$$\therefore \frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$$

$$\sum_{k=1}^{n}\left[\frac{1}{k}+\frac{1}{k+1}\right]=\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\frac{1}{k+1}$$

$$=\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right]+\left[\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\frac{1}{n+1}\right]$$

$$=H_n+H_n-1+\frac{1}{n+1}$$

$$=2H_n-\frac{n}{n+1}$$

(b) $$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)} \delta k$$

$$\sum u\Delta v = uv - \sum \mathrm{E}v\Delta u$$

$$Let \;\; u(k) = 2k+1; \;\; \Delta u(k) = 2;$$

$$\Delta v(k) = \frac{1}{k(k+1)} = (k-1)^{\underline{-2}}$$

$$\left(\text{Since } \Delta(x^{\underline{m}}) = mx^{\underline{m-1}}\right)$$

$$v(k) = -\left(k-1\right)^{\underline{-1}} = -\frac{1}{k}$$

$$Ev = -\frac{1}{k+1}$$

$$\sum \frac{2k+1}{k(k+1)}\delta k = (2k+1)\left(-\frac{1}{k}\right) - \sum\left(-\frac{1}{k+1}\right)2\delta k$$

$$= 2\sum (k^{-1}\delta k) - \frac{2k+1}{k}$$

$$= 2H_k - 2 - \frac{1}{k} + c$$

$$\left[\sum x^{m}\delta x = H_x, \text{if m} = \text{-1[Equation 2.53]}\right]$$

$$\sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)} \delta k = 2H_k - 2 - \frac{1}{k} + c \Big|_1^{n+1}$$

$$= \left[2H_{n+1} - 2 - \frac{1}{n+1} + c\right] - \left[2H_1 - 2 - 1 + c\right]$$

$$= 2H_n + \frac{2}{n+1} - 2 - \frac{1}{n+1} - 2 + 2 + 1$$

$$= 2H_n + \frac{1}{n+1} - 1$$

$$= 2H_n - \frac{n}{n+1}$$