

CSE547

Chapter 2 Problems 27, 29

Problem 27

Question : Compute $\Delta(c^{\frac{x}{k}})$, and use it to deduce the value of

$$\sum_{k=1}^n ((-2)^k / k)$$

Computation of $\Delta(c^x)$

We know that (from class notes) ,

$$\Delta f(x) = f(x+1) - f(x) \quad \text{and}$$

$$x^m = \underbrace{x(x-1)(x-2) \dots (x-m+1)}_{m \text{ factors}}, \quad \text{integer } m \geq 0 \quad (2.43)$$

Computation of $\Delta(c^x)$

Using the formulae given in the previous slide,

$$\Delta(c^x) = c^{x+1} - c^x \quad (\text{Equation I})$$

$$c^{x+1} = \underbrace{c(c-1)(c-2) \dots (c-x)}_{(x+1) \text{ factors}}$$

$$c^x = \underbrace{c(c-1)(c-2) \dots (c-x+1)}_{x \text{ factors}}$$

Computation of $\Delta(c^x)$

Substituting values of c^{x+1} and c^x in equation(I) we get,

$$\Delta(c^x) = (c(c-1)(c-2)\dots(c-x+1)(c-x)) - (c(c-1)(c-2)\dots(c-x+1))$$

Taking $c(c-1)(c-2) \dots (c-x+1)$ common,

$$\begin{aligned}\Delta(c^x) &= c(c-1)(c-2)\dots(c-x+1) ((c-x) - 1) \\ &= c(c-1)(c-2)\dots(c-x+1) \underset{\uparrow}{(c-x-1)} \quad (\text{Equation II})\end{aligned}$$

$(c-x)$ term is missing for c^{x+2}

Computation of $\Delta(c^x)$

Hence, multiplying and dividing equation(II) by $(c-x)$ we get,

$$\Delta(c^x) = \frac{c(c-1)(c-2)\dots(c-x+1) \mathbf{(c-x)}(c-x-1)}{\mathbf{(c-x)}}$$

By definition,

$$c^{\underline{x+2}} = c(c-1)(c-2)\dots(c-x-1)$$

$$\therefore \Delta(c^x) = c^{\underline{x+2}}/(c-x)$$

(Equation III)

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

In order to deduce the value of $\sum_{k=1}^n ((-2)^k/k)$ using the calculated value of $\Delta(c^x)$,

We substitute $c = -2$ and $x = x-2$ in equation(II)

$$\begin{aligned}\Delta((-2)^{x-2}) &= \frac{(-2)^{(x-2)+2}}{((-2)-(x-2))} \\ &= \frac{(-2)^x}{-x} \quad \text{(Equation IV)}\end{aligned}$$

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

Extra Fact:

$$\begin{aligned} -\Delta f(x) &= -(f(x+1) - f(x)) \\ &= f(x) - f(x+1) \\ &= -f(x+1) - (-f(x)) \\ &= \Delta(-f(x)) \end{aligned}$$

$$\therefore \Delta(-(-2)^{x-2}) = \frac{(-2)^x}{x} \quad (\text{Equation V})$$

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

$$\sum_a^b g(x)\delta x = \sum_{k=a}^{b-1} g(k) \quad \text{for integers } b \geq a \quad (2.48)$$

Since, $\sum_{k=1}^n ((-2)^k/k)$ is of the form

$\sum_{k=a}^{b-1} g(k)$ we get,

$$\sum_{k=1}^n ((-2)^k/k) = \sum_{k=1}^{n+1} \frac{(-2)^k}{k} \delta k, \quad \text{for } n \geq 0 \quad (\text{Equation VI})$$

We know that,

$$g(x) = \Delta f(x) \quad \text{iff} \quad \sum g(x) \delta x = f(x) + c \quad (2.46)$$

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

Using Equations IV, VI and 2.46, we get

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-2)^k}{k} \delta k &= -(-2)^{k-2} \Big|_1^{n+1} \\ &= -(-2)^{n+1-2} - (-(-2)^{1-2}) \\ &= -(-2)^{n-1} - (-(-2)^{-1}) \\ &= -(-2)^{n-1} + (-2)^{-1} \\ &= (-2)^{-1} - (-2)^{n-1} \quad (\text{Equation VII})\end{aligned}$$

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

We know that,

$$x^{-m} = \frac{1}{(x+1)(x+2)\dots(x+m)} \quad \text{for } m > 0 \quad (2.51)$$

$$\therefore x^{-1} = 1/(x+1)$$

$$\begin{aligned} \Rightarrow (-2)^{-1} &= 1/(-2+1) \\ &= 1/(-1) \\ &= -1 \end{aligned}$$

Deducing value of $\sum_{k=1}^n ((-2)^k/k)$

Substituting in equation (VII) we get,

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-2)^k}{k} \delta k &= -1 - (-2)^{n-1} \\ &= -1 - ((-2)(-2-1)(-2-2)\dots(-2-(n-2))) \\ &= -1 - ((-2)(-3)(-4)\dots(-n)) \\ &= -1 + ((-1)(-2)(-3)\dots(-n)) \\ &= -1 + (-1)^n n!\end{aligned}$$

Verification

$$\sum_{k=1}^n ((-2)^k / k) = -1 + (-1)^n n!$$

n	1	2	3	4
$\sum_{k=1}^n ((-2)^k / k)$	-2	1	-7	23
$-1 + (-1)^n n!$	-2	1	-7	23

Question 29 of chapter 2

Evaluate

$$\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$$

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

$$\frac{k}{4k^2 - 1} = \frac{k}{(2k)^2 - (1)^2}$$

$$= \frac{k}{(2k - 1)(2k + 1)}$$

Using partial fractions,

$$\frac{k}{(2k - 1)(2k + 1)} = \frac{A}{(2k - 1)} + \frac{B}{(2k + 1)} \quad (\text{eqn I})$$

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

Multiplying and dividing by $(2k - 1) * (2k + 1)$
we get

$$k = A (2k + 1) + B (2k - 1)$$

Grouping powers of k,

$$k + 0 = ((2A) k) + A + ((2B) k) - B$$

Equating powers of k on both sides, we get the
linear equations:

$$2A + 2B = 1 \text{ and} \quad (\text{equation 1})$$

$$A - B = 0 \quad (\text{equation 2})$$

$$\text{Evaluating } \sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$$

Solving the simultaneous equations obtained in the previous slide , we get

$$2A + 2B = 1$$

$$2A - 2B = 0 \quad (\text{multiplying equation 2 by 2})$$

$$4A = 1 \rightarrow A = 1/4$$

$$\text{From equation 2, } B = 1/4$$

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

$$\therefore \frac{k}{4k^2 - 1} = \frac{1}{4} \left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

$$\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)} = \sum_{k=1}^n (-1)^k \frac{1}{4} \left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

We can split the sum on the right side into two summations as follows :

$$\sum_{k=1}^n \frac{(-1)^k}{4} \frac{1}{(2k-1)} + \sum_{k=1}^n \frac{(-1)^k}{4} \frac{1}{(2k+1)}$$

This can be changed to a harmonic sum by putting $2k-1 = m$ and $2k+1 = m$ but that would make it complex.

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\begin{aligned} & \frac{(-1)^1 1}{4} \left(\frac{1}{1} + \frac{1}{3} \right) + \frac{(-1)^2 1}{4} \left(\frac{1}{3} + \frac{1}{5} \right) \\ & + \dots \\ & + \frac{(-1)^n 1}{4} \left(\frac{1}{(2n-1)} + \frac{1}{(2n+1)} \right) \end{aligned}$$

Evaluating $\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\begin{aligned}
 & \frac{1}{4} \left(\frac{(-1)}{1} + \frac{(-1)}{3} \right) + \frac{1}{4} \left(\frac{1}{3} + \frac{1}{5} \right) \\
 & + \frac{1}{4} \left(\frac{(-1)}{5} + \frac{(-1)}{7} \right) + \dots + \\
 & \frac{(-1)^n 1}{4} \left(\frac{1}{(2n-1)} + \frac{1}{(2n+1)} \right)
 \end{aligned}$$

$$\text{Evaluating } \sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)}$$

We can see that the alternate terms get cancelled leaving the first and the last term alone.

$$\frac{-1}{4} + \frac{(-1)^n}{4} \left(\frac{1}{(2n+1)} \right)$$

Which is the required solution.

Verification

$$\sum_{k=1}^n \frac{(-1)^k k}{(4k^2 - 1)} = \frac{-1}{4} + \frac{(-1)^n}{4} \left(\frac{1}{(2n + 1)} \right)$$

n	1	2	3	4
$\sum_{k=1}^n (((-1)^k k)/(4k^2-1))$	-1/3	-1/5	-2/7	-2/9
$(-1/4)+(((1)^n/4)(1/(2n+1)))$	-1/3	-1/5	-2/7	-2/9