CSE547

Chapter 2
Problem 29

THE PROBLEM

EVALUATE:

$$\sum_{k=1}^{n} \frac{(-1)^k\, k}{4k^2 - 1} \qquad \text{........(S1)}$$

OUTLINE OF SOLUTION

- The expression after the $\sum$ sign is called the summand.
- There might exist different ways to evaluate the above given sum.
- One way is to look carefully at the given summand, and see that we can use simple algebra to factorize the denominator and then express the summand as the sum of partial fractions.
- After that we will see that how easy it is to evaluate the sum.

STEP 1 (FACTORING)

We are given the sum:

$$\sum_{k=1}^{n} (-1)^k \, k \;/\; (4k^2 - 1)$$

- Our first step is to factorize the denominator of the summand.
- We know the famous formula:

$$a^2 - b^2 = (a - b)(a + b) \;\ldots\ldots \text{formu}(1)$$

(Where $a, b \in \mathbf{R}$)

STEP 1, Factoring (Cont.)

- So, using formu(1) the denominator of the summand is factorized to :

$$4k^2 - 1 = (2k)^2 - (1)^2$$
$$= (2k - 1)(2k+1)$$

- So, now (s1) becomes:

$$\sum_{k=1}^{n} (-1)^k k \ / \ ((2k - 1)(2k+1)) \ \ldots\ldots \text{sum2}$$

What does the Partial Fraction Theorem of Algebra say?

- Let P (x) and Q (x), be two polynomials, i.e., $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ and
  $Q(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0$
  where $x \in R$, and $a_n \ldots a_0$, $b_m \ldots b_0$ are constants.
- If the degree of $P(x) < Q(x)$ and if we we can factorize Q(x), i.e.,
  $Q(x) = (c_1 x + d_1)(c_2 x + d_2) \ldots (c_k x + d_k)$
  such that no factor is repeated, THEN:

Partial Fraction Theorem (cont)

- There exits constants $A_1, A_2, \ldots\ldots, A_k \in \mathbf{R}$, such that:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(c_1x + d_1)} + \frac{A_2}{(c_2x + d_2)} + \ldots + \frac{A_k}{(c_kx + d_k)}$$

Defined only for those values of x with which $Q(x) \neq 0$

STEP 2, (Using Partial Fraction Decomposition)

Now back to our sum:

$$\sum_{k=1}^{n} (-1)^k \; k \; / ((2k - 1)(2k+1))$$

- Now let us look at . We can see that we can simplify this expression by using partial fraction decomposition. We can use partial fraction decomposition because the degree of the numerator ("k" has degree 1), is less than the degree of the denominator $((2k - 1)(2k+1) = 4k^2 - 1$ has degree 2.) Also, the denominator is product of DISTINCT LINEAR factors.

PARTIAL FRACTION DECOMPOSITION (CONT.)

So, we can simplify k/((2k - 1) (2k+1)) in the following way:

$$\frac{k}{(2k-1)(2k+1)} = \frac{A_1}{(2k-1)} + \frac{A_2}{(2k+1)}$$

We need to now find A_1 and A_2:

$$k = \frac{A_1\cancel{(2k-1)}(2k+1)}{\cancel{(2k-1)}} + \frac{A_2(2k-1)\cancel{(2k+1)}}{\cancel{(2k+1)}}$$

SIMPLIFICATION (CONT)

$$k = A_1\ (2k+1) + A_2(2k-1)$$
$$= 2kA_1 + A_1 + 2kA_2 - A_2$$
$$= 2kA_1 + 2kA_2 + A_1 - A_2$$
$$= (2A_1 + 2A_2)k + A_1 - A_2$$

So now this implies that:

$$2A_1 + 2A_2 = 1 \quad \ldots .\ \text{Eq}(1)\ \&$$
$$A_1 - A_2 = 0 \quad \ldots .\ \text{Eq}(2)$$

So, we have a system of equations that we need to solve.

SIMPLIFICATION (CONT).

- From Eq(2) , we know that

$$A_1 = A_2$$

- Now substituting A_1 instead of A_2 in Eq(1) we get:

$$2A_1 + 2A_2 = 1$$
$$2A_1 + 2A_1 = 1$$
$$4A_1 = 1$$
$$A_1 = ¼$$

- Since $A_1 = A_2$, therefore $A_2 = ¼$

So, A1 = A2 = ¼

$$\frac{k}{(2k-1)(2k+1)} = \frac{A_1}{(2k-1)} + \frac{A_2}{(2k+1)}$$

$$\frac{k}{(2k-1)(2k+1)} = \frac{1}{4(2k-1)} + \frac{1}{4(2k+1)}$$

So, we had:

$$\sum_{k=1}^{n} (-1)^k k \;/\; (4k^2 - 1)$$

$$= \sum_{k=1}^{n} (-1)^k k \;/\; ((2k-1)(2k+1))$$

$$= \sum_{k=1}^{n} ((-1)^k (\frac{1}{4(2k-1)} + \frac{1}{4(2k+1)}))$$

Continuing Evaluation of Sum

$$=1/4 \sum_{k=1}^{n} ((-1)^k (\frac{1}{(2k-1)} + \frac{1}{(2k+1)}))$$

$$=1/4 \sum_{k=1}^{n} (\frac{(-1)^k}{(2k-1)} + \frac{(-1)^k}{(2k+1)})$$

$$=1/4 (\sum_{k=1}^{n} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k+1})$$

$$=1/4 (\frac{(-1)}{2(1)-1} + \sum_{k=2}^{n} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k+1})$$

Continuing Evaluation of Sum

$$=1/4\left(-1+\sum_{k=2}^{n}\frac{(-1)^k}{2k-1}+\sum_{k=1}^{n}\frac{(-1)^k}{2k+1}\right)$$

$$=1/4\left(-1+\left(\sum_{k=2}^{n}\frac{(-1)^k}{2k-1}\right)+\left(\sum_{k=1}^{n-1}\frac{(-1)^k}{2k+1}\right)+\frac{(-1)^n}{2n+1}\right)$$

LET US CONCENTRATE ON THIS SUM.
OUR GOAL IS TO NOW SHOW THAT THIS SUM IS THE NEGATION OF THAT SUM.
AND THUS THE TWO SUMS ARE GOING TO CANCEL OUT.

Continuing Evaluation of Sum

Let us concentrate on the first sum:

$$\sum_{k=2}^{n} \frac{(-1)^k}{2k - 1} = \sum_{2 \le k \le n} \frac{(-1)^k}{2k - 1}$$

$$= \sum_{2 \le k+1 \le n} \frac{(-1)^{k+1}}{2(k+1) - 1}$$

$$= \sum_{2 \le k+1 \le n} \frac{(-1)^{k+1}}{2k+2 - 1}$$

$$= \sum_{2 \le k+1 \le n} \frac{(-1)(-1)^k}{2k+1}$$

Continuing Evaluation of Sum

$$\sum_{k=2}^{n} \frac{(-1)^k}{2k-1} = (-1) \sum_{2 \le k+1 \le n} \frac{(-1)^K}{2k+1}$$

$$= - \sum_{1 \le k \le n-1} \frac{(-1)^K}{2k+1}$$

$$= - \sum_{k=1}^{n-1} \frac{(-1)^K}{2k+1}$$

$$\boxed{\sum_{k=2}^{n} \frac{(-1)^k}{2k-1} = - \sum_{k=1}^{n-1} \frac{(-1)^K}{2k+1} \qquad \text{......(info2)}}$$

Continuing Evaluation of Sum

We had shown that:

$$\sum_{k=1}^{n} (-1)^k \, k \;/\; (4k^2 - 1)$$

$$=1/4\left(-1+\left(\sum_{k=2}^{n} \frac{(-1)^k}{2k-1}\right)+\left(\sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1}\right)+\frac{(-1)^n}{2n+1}\right)$$

(Now using info 2)

$$=1/4\left(-1-\cancel{\sum_{k=1}^{n-1}\frac{(-1)^k}{2k+1}}+\cancel{\sum_{k=1}^{n-1}\frac{(-1)^k}{2k+1}}+\frac{(-1)^n}{2n+1}\right)$$

Continuing Evaluation of Sum

$$\sum_{k=1}^{n} (-1)^k \, k \;/\; (4k^2 - 1)$$

$$= \tfrac{1}{4}\left(-1 + \frac{(-1)^n}{2n+1}\right)$$

$$= \frac{-1}{4} + \frac{(-1)^n}{8n+4}$$

$$\boxed{\sum_{k=1}^{n} (-1)^k \, k \;/\; (4k^2 - 1) = \frac{-1}{4} + \frac{(-1)^n}{8n+4}}$$

(THE END !)

Observation, Intuition and then Generalization

We had the following information:

SUM 1 SUM 2

$$\sum_{k=1}^{n} (-1)^k k \ / \ (4k^2 - 1)$$

$$= 1/4\left(\sum_{k=1}^{n} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k + 1} \right)$$

Now, How did we know that the first term of sum1 does not cancel out with any thing and the last term of the sum2 does not cancel out with anything and everything else of the two sums cancel out with one another ?
Well,…, we evaluated the sums for small values of "n" and saw what was happening !

Evaluating the sums for small values of n

So, the expression that we are concentrating on is:

$$\sum_{k=1}^{n} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k + 1}$$

Let n be 1:

$$\sum_{k=1}^{1} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{1} \frac{(-1)^k}{2k + 1} = \frac{-1}{2(1) - 1} + \frac{-1}{2(1) + 1}$$

$$= -1 + \frac{-1}{3}$$

(nothing cancels)

Evaluating the sums for small values of n

Let n be 2:

$$\sum_{k=1}^{2} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{2} \frac{(-1)^k}{2k + 1} = \frac{-1}{2(1) - 1} + \frac{1}{2(2) - 1}$$

$$+ \frac{-1}{2(1) + 1} + \frac{1}{2(2) + 1}$$

$$= -1 + \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} + \frac{1}{5}$$

So, we can see that the first term of the first sum remains and the last term of second sum remains. But the second term of the first sum cancels with the first term of the second sum.

- In other words we see that when n=2:

$$\sum_{k=2}^{2} \frac{(-1)^k}{2k - 1} = - \sum_{k=1}^{1} \frac{(-1)^k}{2k + 1}$$

Evaluating the sums for small values of n

Let n be 3:

$$\sum_{k=1}^{3} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{3} \frac{(-1)^k}{2k+1} = \frac{-1}{2(1)-1} + \frac{1}{2(2)-1} + \frac{-1}{2(3)-1}$$

$$+ \frac{-1}{2(1)+1} + \frac{1}{2(2)+1} + \frac{-1}{2(3)+1}$$

$$= -1 + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{5}}$$

$$- \frac{1}{7}$$

$$= -1 - \frac{1}{7}$$

OBSERVATION

- So, when n =3, we see that again the first term of the first sum does not cancel with anything and the last term of the second sum does not cancel with anything.
- We further see that the second term of sum 1 cancels with the first term of sum2.
- Also, the third term of sum1 cancels with the second term of sum 2.
- Thus everything cancels except the first term of sum 1and the last term of sum 2.

- In other words we see that when n=3:

$$\sum_{k=2}^{3} \frac{(-1)^k}{2k - 1} = - \sum_{k=1}^{2} \frac{(-1)^k}{2k + 1}$$

GENERALIZATION

- Our observation leads us to generalize what happens when we have "n" as the upper limit of the two sums.
- Thus, for:

$$\sum_{k=1}^{n} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k + 1}$$

Let us name this Exp

We can say that the first term of the first sum does not cancel out. So, we separate the first term of the first sum and make the first sum go from k= 2 to n.

GENERALIZATION

We are going to have:

$$\text{Exp}= -1+ \sum_{k=2}^{n} \frac{(-1)^k}{2k - 1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k + 1}$$

Also, we had observed that the last term of the second sum does not cancel with anything so, we separate that term also. We are going to get:

$$\text{Exp}= -1+\left(\sum_{k=2}^{n} \frac{(-1)^k}{2k - 1}\right)+\left(\sum_{k=1}^{n-1} \frac{(-1)^k}{2k + 1}\right)+\frac{(-1)^n}{2n + 1}$$

Now we will prove the general case !

- Now, we have an intuition from our observation of the cases n=1, n=2 and n=3, that the two sums on the last line of the previous slide must cancel out.
- And we start to prove that the two sums cancel out and we succeed to prove this intuition of ours, on slides 15 and 16.