Problem 31
Chapter 2

CSE547

- The Rieman Zeta function is defined as follows:

$$\zeta(k) = \sum_{j=1}^{\infty} \frac{1}{j^k}$$

$$\zeta(1) = \sum_{j=1}^{\infty} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

- $\zeta(1)$ is undefined because the summation diverges. However, $\zeta(2)$, $\zeta(3)$,… are all convergent.

Each of the following are convergent. Their sums, however, are different.

$$\zeta(2) = \sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

$$\zeta(3) = \sum_{j=1}^{\infty} \frac{1}{j^3} = 1 + \frac{1}{8} + \frac{1}{27} + \ldots$$

$$\zeta(4) = \sum_{j=1}^{\infty} \frac{1}{j^4} = 1 + \frac{1}{16} + \frac{1}{81} + \ldots$$

$$\zeta(5) = \sum_{j=1}^{\infty} \frac{1}{j^5} = 1 + \frac{1}{32} + \frac{1}{243} + \ldots$$

Problem #31:

- The problem in the textbook stated:
- 1.) We wish to prove that $\Sigma\ (\zeta(k) - 1) = 1$, for $k \geq 2$.
- 2.) We need to also find out what the following expression is equal to:

$$\Sigma\ (\zeta(2k) - 1) \text{ for } k \geq 1.$$

- Let's start by proving that $\Sigma (\zeta(k) - 1) = 1$, for $k \geq 2$.
- $\zeta(2) - 1 = 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2\ldots$
- $\zeta(3) - 1 = 1/2^3 + 1/3^3 + 1/4^3 + 1/5^3\ldots$
- $\zeta(4) - 1 = 1/2^4 + 1/3^4 + 1/4^4 + 1/5^4\ldots$
- $\zeta(5) - 1 = 1/2^5 + 1/3^5 + 1/4^5 + 1/5^5\ldots$
- . . . .
- . . . .
- If we add along the columns of this array, we get this:
- $(1/2^2 + 1/2^3 + 1/2^4 + 1/2^5 + \ldots) + (1/3^2 + 1/3^3 + 1/3^4 + 1/3^5\ldots) + (1/4^2 + 1/4^3 + 1/4^4 + 1/4^5 \ldots) + \ldots$

- A simplified version of this would be:

  $= \Sigma 1/2^k + \Sigma 1/3^k + \Sigma 1/4^k + \ldots$ (with k going from 1 to ∞)

  $= \Sigma\ (\Sigma 1/n^k)$.

The inner summation ($\Sigma 1/n^k$) is nothing more than a geometric series. It is convergent because the sequence goes to zero as k goes to infinity. To find the sum of a convergent geometric series, we take the first term, a_1, and then divide by (1-r), where r is the common ratio.

Proof:

- Let S be a geometric series with first term 'a_0' and common ratio 'r'. It may be finite or infinite.
- Then $S = a_0 + a_0r + a_0r^2 + \ldots + a_0r^{n-1}$
- Then $r*S = a_0r + a_0r^2 + \ldots + a_0r^{n-1} + a_0r^n$
-
- $S - r*S = a_0 - a_0r^n$
- $S = a_0(1 - r^n) / (1 - r)$
- $S = a_0 / (1 - r)$ if $r^n \rightarrow 0$.
- Therefore, $S = a_0 / (1 - r)$ if r is less than one.

- $\Sigma 1/n^k = (1/n^2) / (1 – 1/n)$
- because 1/n is the first term and n is the common ratio.

$=(1/n^2)*(n / n – 1)$

Now we need to split the equation into two parts. This is called partial fraction decomposition.

$$= \frac{1}{n-1} - \frac{1}{n}$$ because of partial fractions.

- Therefore, $\Sigma (\Sigma 1/n^k) = \Sigma \left(\frac{1}{n-1} - \frac{1}{n}\right)$ where n goes from 2 to infinity.

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

= 1 – ½ + ½ – 1/3 + 1/3 – ¼ + ¼ +…

=1 – ½ + ½ – 1/3 + 1/3 – ¼ + ¼ +… = 1.

Since the terms go to zero, we can cancel every term out with its' additive inverse. Therefore, the sum of the series is one.

We have proved the first problem.

Now, let’s try to find out what the following expression is equal to:

$$\Sigma\ (\zeta(2k) - 1) \text{ for } k \geq 1.$$

We know that this summation should be smaller than one because we are taking only the even zeta numbers.

- Let’s write down some of the series so that we can get an idea of what’s going on:
- $\zeta(2) - 1 = 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + \ldots$
- $\zeta(4) - 1 = 1/2^4 + 1/3^4 + 1/4^4 + 1/5^4 + \ldots$
- $\zeta(6) - 1 = 1/2^6 + 1/3^6 + 1/4^6 + 1/5^6 + \ldots$
- . . . . . . …
- . . . . . . …
- Recall, that each of these series is convergent. Again, we will take each column and try to sum it up and then add all of the columns together at the end.

- $= (1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + \ldots) + (1/2^4 + 1/3^4 + 1/4^4 + 1/5^4 + \ldots) + (1/2^6 + 1/3^6 + 1/4^6 + 1/5^6 + \ldots) + (\ldots)$

- $= (1/2^2 + 1/2^4 + 1/2^6 + \ldots) + (1/3^2 + 1/3^4 + 1/3^6 + \ldots) + (1/4^2 + 1/4^4 + 1/4^6 + \ldots) + (1/5^2 + 1/5^4 + 1/5^6 + \ldots) + \ldots$

- $=\Sigma\ (\Sigma 1/n^{2k})$. Again, the inner summation is just a geometric series that converges, and therefore, we can take the sum very easily.

- $\Sigma 1/n^{2k} = (1/n^2) / (1 - 1/n^2)$
- $= (1/n^2) / \{n^2/ (n^2 - 1)\}$
- $= 1/(n^2 - 1)$.
- Therefore, the expression $\Sigma (\Sigma 1/n^{2k})$ simplifies to
- $\Sigma (1/n^2 - 1)$. If we try to evaluate this, we will need to break this down into partial fractions.
- This leads us to:
- $\Sigma \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} [1 - 1/3 + \frac{1}{2} - \frac{1}{4} + 1/3 - 1/5 \ldots]$
- Because of the telescoping property, this is what happens:
- $= \frac{1}{2} [1 - 1/3 + \frac{1}{2} - \frac{1}{4} + 1/3 - 1/5 \ldots]$
- $= \frac{1}{2} [1 + \frac{1}{2}]$
- $= \frac{3}{4}$. Therefore, the sum is $\frac{3}{4}$.