CSE 547: DISCRETE MATHEMATICS

Chapter 3: Problem 11

Problem Definition

Chapter 3. Problem 11

- Prove that the open interval, (α, β) i.e. $(\alpha \ldots \beta)$ contains $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ integers, where $\alpha < \beta$.
- Also, show why case $\alpha = \beta$ has to be excluded.

Given

- Open Interval ($\alpha \ldots \beta$) i.e. a number γ belongs to the open interval if and only if $\alpha < \gamma < \beta$.
- α, β are real numbers.
- $\alpha < \beta$

Groundwork

We know the following from the definition of the Floor & Ceiling Function

- $x < n \quad \Rightarrow \quad \lfloor x \rfloor < n$ ....(1)
- $n < x \quad \Rightarrow \quad n < \lceil x \rceil$ ....(2)

Where, x is a real number and n is an integer.

Groundwork

$x < n \Rightarrow \lfloor x \rfloor < n$	$n < x \Rightarrow n < \lceil x \rceil$

$x < n$ ……(given)

$\lfloor x \rfloor \leq x$ ……(by defn)

$\Rightarrow \lfloor x \rfloor < n$

$\therefore\ x < n \Rightarrow \lfloor x \rfloor < n$

$n < x$ ……(given)

$\lceil x \rceil \geq x$ ……(by defn)

$\Rightarrow n < \lceil x \rceil$

$\therefore\ n < x \Rightarrow n < \lceil x \rceil$

Groundwork

- Consider a list of integers,

  a, a+1, a+2, … b–2, b–1, b

- The number of integers in the above list is given by,

  #integers = (b-a) + 1 ………….(3)

- Example: 3, 4, 5, 6, 7, 8

  a = 3, b = 8

  #integers = (b-a) + 1 = (8-3) + 1 = 6

Proof

Now we have all necessary facts to prove our formula.

Proof

- Open Interval $(\alpha \ldots \beta)$, α, β are real
- $\alpha < \beta$
- Let n be an integer such that,

  $\alpha < n < \beta$ ....(4)
- Breaking up (4),

  $\alpha < n$ & $n < \beta$

  where, α, β are real and n is an integer.

Proof

- Consider $\alpha < n$

  Using (1),

  $\alpha < n \Rightarrow \lfloor \alpha \rfloor < n$ ……(5)

- Consider $n < \beta$

  Using (2),

  $n < \beta \Rightarrow n < \lceil \beta \rceil$ ……(6)

Proof

- Combining inequalities (5) and (6),

  $\alpha < n < \beta \Rightarrow \lfloor \alpha \rfloor < n < \lceil \beta \rceil$ ......(7)

- Thus the inequality $\alpha < n < \beta$ reduces to,

  $\lfloor \alpha \rfloor < n < \lceil \beta \rceil$,

  where now, $\lfloor \alpha \rfloor$, n, $\lceil \beta \rceil$ are all integers.

Proof

- Observing $\lfloor\alpha\rfloor < n < \lceil\beta\rceil$, we come to know that, n can take the following values,

$$\underbrace{\lfloor\alpha\rfloor+1, \lfloor\alpha\rfloor+2 \ldots \lceil\beta\rceil-3, \lceil\beta\rceil-2, \lceil\beta\rceil-1}_{\text{List of Integers}}$$

- Above is a list of integers with,

$a = \lfloor\alpha\rfloor+1$

$b = \lceil\beta\rceil-1$

Proof

- Using (3),

$$\#\text{integers} = (b-a) + 1$$
$$= \{\lceil \beta \rceil - 1\} - \{\lfloor \alpha \rfloor + 1\} + 1$$
$$= \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 - 1 + 1$$
$$= \lceil \beta \rceil - \lfloor \alpha \rfloor - 1$$

Proof

$\therefore$ #integers in $\lfloor \alpha \rfloor < n < \lceil \beta \rceil$ is $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

i.e. #integers in $\alpha < n < \beta$ is $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

$\therefore$ #integers in the interval $(\alpha \ldots \beta) = \lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

Hence Proved!

Example

- (2.3…8.5)
  $\alpha = 2.3$
  $\beta = 8.5$
  $\lfloor \alpha \rfloor = 2$
  $\lceil \beta \rceil = 9$

  #integers in (2.3…8.5) $= \lceil \beta \rceil - \lfloor \alpha \rfloor - 1$
  $= 9 - 2 - 1$
  $= 6$

  Enumerating the integers: 3, 4, 5, 6, 7, 8 i.e. 6 integers.

Special Case

Case: $\alpha = \beta$

- Does the formula work when $\alpha = \beta$?

  When $\alpha = \beta$,

  $\lfloor \alpha \rfloor = \lfloor \beta \rfloor$ & $\lceil \alpha \rceil = \lceil \beta \rceil$

- Example: $\alpha = \beta = 5.6$

  $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = 5$

  $\lceil \alpha \rceil = \lceil \beta \rceil = 6$

  #integers in (5.6…5.6) = 6-5-1 = 0 (Right)

Case: $\alpha = \beta$

- The formula seems to work fine when $\alpha = \beta$.
- Now let us consider that the special case that, $\alpha = \beta$ is an integer.
- In that case,
  $\lfloor \alpha \rfloor = \lfloor \beta \rfloor = \lceil \alpha \rceil = \lceil \beta \rceil$
  $\therefore$ We will get the incorrect answer,
  #integers in $(\alpha \ldots \beta) = \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 = -1$ (Wrong)
- Thus, Case: $\alpha = \beta$ is an integer needs to be excluded while using the derived formula.

Summary

- Open Interval $(\alpha \ldots \beta)$
- α, β are real numbers

$$\#\text{integers in } (\alpha \ldots \beta) = \begin{cases} 0, & \alpha = \beta \text{ is an integer} \\ \lceil \beta \rceil - \lfloor \alpha \rfloor - 1, & \text{otherwise} \end{cases}$$