CSE547 Discrete Mathematics

Chapter 3: Problem 17

Problem Statement

Evaluate the sum $\sum_{0 \le k < m} \lfloor x + k/m \rfloor$ in the case $x \ge 0$ by substituting $\sum_j [1 \le j \le x + k/m]$ for $\lfloor x+k/m \rfloor$ and summing first on k. Does your answer agree with (3.26)?

Equation (3.26) is

$$\lfloor mx \rfloor = \lfloor x \rfloor + \lfloor x + 1/m \rfloor + \ldots\ldots + \lfloor x + (m-1)/m \rfloor$$

Our problem statement now becomes

$$\sum_{j,k} [0 \le k < m][1 \le j \le x + k/m], \quad x \ge 0$$

Splitting up we get

$$\sum_{j,k} [0 \le k < m][1 \le j \le x + k/m] =$$

$$\sum_{j,k} [0 \le k < m][1 \le j \le \lceil x \rceil][k \ge m(j - x)]$$

$$\left(\because\ j \le x + k/m \text{ gives } m(j-x) \le k\right) \text{ and}$$

$$\left(j \le x + k/m,\ j - x \le k/m,\ j - x < 1,\ j < x + 1 \text{ so } j \le \lceil x \rceil\right)$$

We have to sum first on k

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_{k} [0 \le k < m][k \ge m(j - x)]$$

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_{k} [0 \le k < m] - \sum_{1 \le j \le \lceil x \rceil} \sum_{k} [0 \le k < m(j - x)] \qquad (1)$$

For the limit of j in the second summation

for $1 \le j < \lceil x \rceil$, $\quad m(j - x) < 0$

$\therefore \; j = \lceil x \rceil$ is the limit for j

So now equation (1) can be wriiten as

$$= \sum_{1 \le j \le \lceil x \rceil} \sum_k [0 \le k < m] - \sum_{j = \lceil x \rceil} \sum_k [0 \le k < m(j-x)]$$

$$= \sum_{1 \le j \le \lceil x \rceil} m - \sum_{j = \lceil x \rceil} \lceil m(j-x) \rceil$$

$$\left(\because \; [\alpha \,..\, \beta) \text{ contains } \lceil \beta \rceil - \lceil \alpha \rceil \text{ integers} \right) \qquad (3.12)$$

$$= m \sum_{1 \le j \le \lceil x \rceil} 1 - \sum_{j = \lceil x \rceil} \lceil m(j-x) \rceil$$

Substituting $j = \lceil x \rceil$

$$= m(\lceil x \rceil - \lceil 1 \rceil + 1) - \lceil m(\lceil x \rceil - x) \rceil$$

$(\because [\alpha..\beta]$ contains $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1)$ and (3.12)

$(\because \lfloor \lceil x \rceil \rfloor = \lceil x \rceil)$

$$= m \lceil x \rceil - m \lceil x \rceil - \lceil - mx \rceil$$

$(\because \lceil \lceil \mathrm{x} \rceil \rceil = \lceil x \rceil, \mathrm{x} \geq 0)$

$$= -\lceil -mx \rceil$$

Using the reflections property $\lceil -x \rceil = -\lfloor x \rfloor$

which gives

$$= \lfloor mx \rfloor$$

This is equation (3.26)

Hence Proved