Discrete Mathematics
CSE 547

Chapter 3: Problems 33, 36

PROBLEM 36

Assuming that ‘n’ is a non-negative integer, find a closed form for the sum

$$\sum_{1 < k < 2^{2^n}} \left[1/ \left(2^{\lfloor \lg k \rfloor} 4^{\lfloor \lg \lg k \rfloor}\right) \right]$$

Let S = $\sum_{1 < k < 2^{2^n}} [\, 1/\,(2^{\lfloor \lg k \rfloor}\, 4^{\lfloor \lg \lg k \rfloor}\,)\,]$

Solution:

We can reduce the problem to a simpler form by replacing the power factors by simple notations thus,
We assume that lg k means $\log_2 k$
Let $l = \lfloor \lg k \rfloor$, $m = \lfloor \lg l \rfloor$ ——— > Eqn (1)

The problem reduces to:

$$S = \sum_{l,m,1 < k < 2^{2^n}} [\, 1/\,(2^{l}\, 4^{m}\,)\,] =$$

$$\sum_{l,m,1 < k < 2^{2^n}} (2^{-l}\, 4^{-m}\,)$$

——— > Eqn (2)

From Eqn 2, we can deduce,

$$S = \sum_{l,\, m,\, 1 < k < 2^{2^n}} (2^{-l}\, 4^{-m})$$

From Eqn 1, we have $l = \lfloor \lg k \rfloor$, $m = \lfloor \lg l \rfloor$

We need to bind the values of l, m and k such that the summation can be easily split and evaluated.

From Eqn 1,

$l = \lfloor \lg k \rfloor$

$==> l \leq \lg k < l + 1$, From Eqn 3.5 (a)

Taking anti-log on both sides of the inequality,

$\boxed{2^l \leq k < 2^{l+1}}$ ----> Bounds for k

Thus we find out the bounds of k in terms of l.

Similarly we need to bound the value l.

From Eqn 1, we know that,

$m = \lfloor \lg l \rfloor$

$\Longrightarrow m \leq \lg l < m + 1$, From Eqn 3.5 (a)

Taking anti-log on all sides of the inequality,

$\boxed{2^m \leq l < 2^{m+1}}$ --- > Bounds for l

From Eqn 1, we know that, k also lies in $(1, 2^{2^n})$

$==> 1 < k < 2^{2^n}$

We know that $l = \lfloor \lg k \rfloor$

For $k > 1$, $l > 0$,

$k = 2$, $l = 1$

$k < 2^{2^n}$, taking log on both sides of the inequality, we get

$\lg k < 2^n$ From Eqn 1, $l < \lg k$

$l < 2^n$ and $l \geq 1 ==> 1 \leq l < 2^n$ ---- > Eqn - 3

From Eqn 3,

$1 \le l < 2^n$

From Eqn 1, we know that,

$m = \lfloor \lg l \rfloor \implies m < \lg l$

$\implies$ For $l = 1$, $m = 0$,

$l < 2^n$

Taking log on both sides,

$\lg l < n \implies m < n$

$\implies \boxed{0 \le m < n}$ ---> Bounds for m

By applying the bounds for l, m and k in Eqn 2, we get

$$S = \sum_{l,\, m,\, 1 < k < 2^{2^n}} (2^{-l}\, 4^{-m})$$

$$S = \sum_{2^m \leq l < 2^{m+1},\, 2^l \leq k < 2^{l+1},\, 0 \leq m < n} (2^{-l}\, 4^{-m})$$

By definition of multiple sums, we can re-write the above equation as,

$$S = \sum_{2^m \leq l < 2^{m+1}} \sum_{2^l \leq k < 2^{l+1}} \sum_{0 \leq m < n} (2^{-l}\, 4^{-m})$$

$$S = \sum_{2^m \le l < 2^{m+1}} \sum_{2^l \le k < 2^{l+1}} \sum_{0 \le m < n} (2^{-l} 4^{-m})$$

We can interchange the order of summations by applying the associative law of summations.

$$S = \sum_{2^m \le l < 2^{m+1}} \sum_{0 \le m < n} \sum_{2^l \le k < 2^{l+1}} (2^{-l} 4^{-m})$$

$$S = \sum_{0 \le m < n} (4^{-m}) \sum_{2^m \le l < 2^{m+1}} (2^{-l}) \sum_{2^l \le k < 2^{l+1}}$$

- -- > Eqn 4

Let $S_1 = \sum_{2^l \leq k < 2^{l+1}} (1)$ in Eqn 4

$S_1 = 2^{l+1} - 2^l$

$S_1 = 2^l * (2 – 1)$

$S_1 = 2^l$ --- > Inference (1)

Using the above inference in Eqn 4

$$S = \sum_{0 \leq m < n} (4^{-m}) \sum_{2^m \leq l < 2^{m+1}} (2^{-l}) * S_1$$

$$S = \sum_{0 \leq m < n} (4^{-m}) \sum_{2^m \leq l < 2^{m+1}} (2^{-l}) * 2^l$$

$$S = \sum_{0 \le m < n} (4^{-m}) \sum_{2^m \le l < 2^{m+1}} (2^{-l}) * 2^l$$

$$S = \sum_{0 \le m < n} (4^{-m}) \sum_{2^m \le l < 2^{m+1}} (1) \text{ --- > Eqn } 5$$

Let $S_2 = \sum_{2^m \le l < 2^{m+1}} (1)$ in the above equation

$S_2 = 2^{m+1} - 2^m$

$S_2 = 2^m * (2 - 1)$

$S_2 = 2^m$ --- > Inference (2)

Using the inference (2) in Eqn 5,

$$S = \sum_{0 \leq m < n} 4^{-m} S_2$$

$$S = \sum_{0 \leq m < n} 4^{-m} * 2^{m}$$

$$S = \sum_{0 \leq m < n} 2^{-m} * 2^{-m} * 2^{m}$$

$$S = \sum_{0 \leq m < n} 2^{-m}$$

$S = 1 + (1/2) + (1/2^2) + \dots\dots\dots\dots + (1/2^{(n-1)})$

The above summation is a geometric series summation.

$S = 1 + (1/2) + (1/2^2) + \ldots\ldots\ldots\ldots + (1/2^{(n-1)})$

By applying the geometric summation equation for this with the reminder = (1/2) and initial term = 1 over 'n' terms, we get

$S = 1 * (1 – (1/2)^n) / (1 – (1/2))$

$S = 2 * (1 – 2^{-n})$

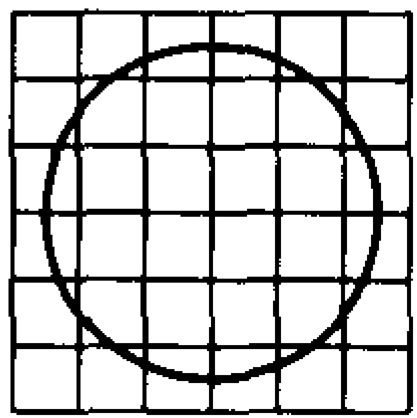
Thus we have got a closed form:

$$\boxed{\mathbf{S = 2(1 – 2^{-n})}}$$

End of Problem - 36

Problem 33

A circle, 2n-1 units in diameter, has been drawn symmetrically on a 2n X 2n chess-board, illustrated here for n = 3.



(a) How many cells of a board contain a segment of the circle ?

(b) Find a function $f(n, k)$ such that exactly $\sum f(n, k)$ $[1 \leq k \leq n - 1]$ cells of the board lie entirely within the circle.

(a) How many cells of a board contain a segment of the circle ?

A segment of a circle is one that is bounded by a chord and an arc.

From the figure, we can observe that the circle does not pass through any of the corners of any of the cells.
But, we need to prove this fact for the general case for any “n”. Let us call this Fact – 1.

To prove: Given diameter = (2n – 1) units, the circle cannot pass through any corner of any cell.

Proof:

Given diameter of the circle = (2n – 1) units

➔ radius, r = (n – ½) units ----- > (1)

For the circle to pass through any corners, we can represent radius "r" in basically 2 ways:

1> r = m units, where m = number of cells that "r" can cover by moving along the sides of the cells. $m \in Z$

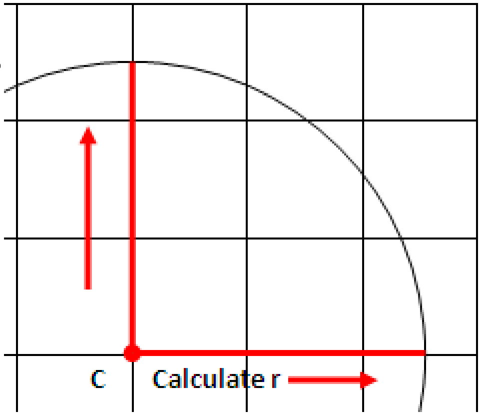
2> r = m*√2 units, where m = number of cells that "r" can cover by moving along the diagonals of the cells. $m \in Z$

We know for sure that, center of the circle, is at a corner of a cell, hence we can deduce the value of 'r' in terms of diagonals of the cells.

From Pythagoras theorem, we know that the length of a diagonal of a cell = $\sqrt{1 + 1} = \sqrt{2}$

Case 1: *$r = m$ units, where m = number of cells that "r" can cover by moving along the sides of the cells. $m \in Z$*

From the above statement, $\{ r, m \} \in Z$



Proof:

From Eqn 1, we know that

$r = (n – ½)$ units

We know that, $n \in Z$ == > r is not an integer.

This contradicts our assumption that r and hence m is an integer.

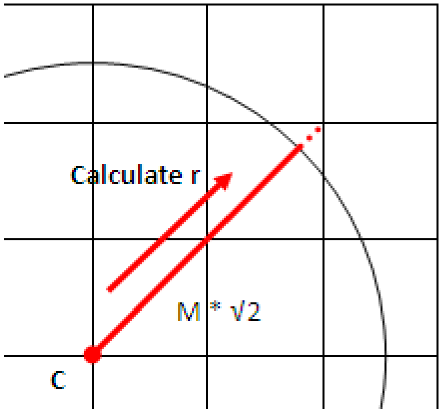
== > r cannot pass through the corner of cells when r is measured in terms of the length of side of each cell.

Case 2: *r = m*√2 units, where m = number of cells that "r" can cover by moving along the diagonals of the cells. m Є Z*

$r = m*\sqrt{2}$

$r^2 = 2*m^2$ == > $r^2 \in Z$

From the above statements, $\{ r^2 , m^2 \} \in Z$



Proof:

From Eqn 1, we know that

$r = (n - ½)$ units , we know that, $n \in Z$

$r^2 = (n - ½)^2$ units, == > r^2 is not an integer.

This contradicts our assumption that r^2 and hence m^2 is an integer.

== > r cannot pass through the diagonals of any cells when r is measured in terms of the length of diagonals of each cell.

Thus we have proved that, the circle cannot pass through any of the corners of any of the cells, for any number of cells “n”.

By proving this, we can claim that the circle has to pass through the cells.

From the figure, we can observe that we can attribute one cell for every intersection point of the circle with the cells.

== > *number of intersection points =*

number of cells that the circle passes through.

-- >Eqn (2)

The circle cuts $(2n - 1)$ horizontal lines, and $(2n - 1)$ vertical lines.
Each such line is cut twice.

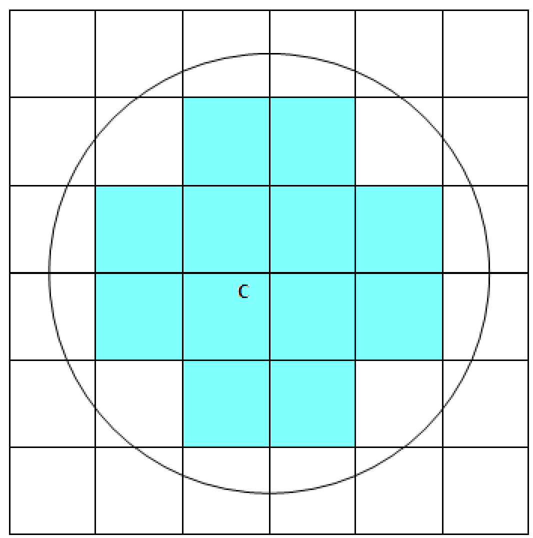
== > number of intersection points

$$= 2 * (2n - 1) + 2 * (2n - 1)$$
$$= 4n - 2 + 4n - 2$$
$$= 8n - 4$$

From Eqn (2),
Number of cells that the circle passes through = $8n - 4$
A segment of the circle is one bounded by a chord and an arc.
From the figure, we can observe that all the cells that the circle cuts are sure to contain atleast one segment of the circle.

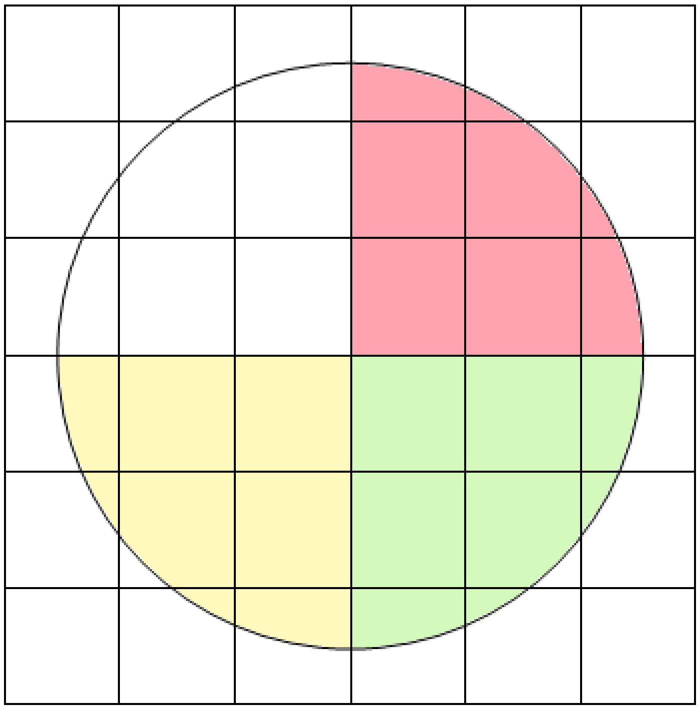
== > Number of cells that contain a segment of the circle = $8n - 4$

(b) Find a function f (n, k) such that exactly $\sum f(n, k)\ [1 \le k \le n - 1]$ cells of the board lie entirely within the circle.



Since the circle is drawn symmetrically inside the chessboard, all four sectors contain equal number of full squares.

Therefore it is enough if we calculate the number of cells that lie inside one sector and that can be multiplied by 4 to get the final result.



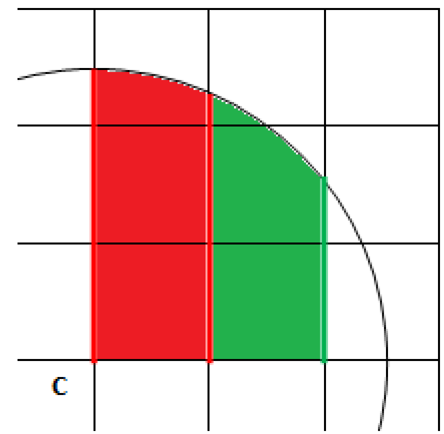
Goal : To calculate the number of cells that lie entirely inside a sector

We know that the circle cuts 2n-1 verticals lines

Now, excluding the vertical line that passes through the center of circle , we have 2n-2 vertical lines that the circle cuts.

Therefore the number of vertical lines cut by a sector is given by,

$$(2n - 2)/\ 2 = n - 1$$



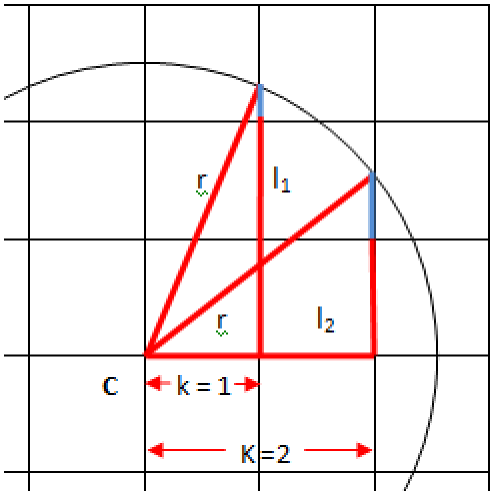
This means a sector has n-1 vertical strips of unit width passing through it.
Also we can observe that all the squares a sector encompasses are part of the vertical strips.

Length of the vertical line of a strip inside a sector can be calculated using Pythagoras theorem as below,

$l_1 = \sqrt{(r^2 - 1^2)}$

$l_2 = \sqrt{(r^2 - 2^2)}$

Now it can be easily observed from the fig. that $\lfloor l_1 \rfloor$ and $\lfloor l_2 \rfloor$ give the number of full squares in the vertical strip that are part of the strip.



If we sum up the number of such full squares of the vertical strips, we get,

No. of squares in a sector =

$$\sum \lfloor \sqrt{(r^2 - k^2)} \rfloor \quad [1 \le k \le n - 1]$$

Therefore the number of squares that lie entirely inside the circle is given by,

$$\sum f(n,k) \ [1 \le k \le n - 1] \quad =$$

$$\sum (4 * \lfloor \sqrt{(r^2 - k^2)} \rfloor) \ [1 \le k \le n - 1]$$

$$\Longrightarrow f(n,k) = 4 * \lfloor \sqrt{(r^2 - k^2)} \rfloor$$