CSE547

Chapter 3, Problem 35

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Simplify the formula

$$\left\lfloor (n+1)^2 n! e \right\rfloor \bmod n \qquad n \text{ is an integer, where } n >= 0$$

First we can just look at the term inside the falling power

$$(n+1)^2 n! e$$

We know

$$e = \sum_{k>=0} \frac{1}{k!}$$

$$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$(n+1)^2 n!e = \frac{(n+1)^2 n!}{0!} + \frac{(n+1)^2 n!}{1!} + \frac{(n+1)^2 n!}{2!} + \cdots$$

$$= \frac{(n+1)^2 n!}{0!} + \frac{(n+1)^2 n!}{1!} + \cdots + \frac{(n+1)^2 n!}{(n-1)!}$$

A_n

$$+ \frac{(n+1)^2 n!}{n!} + \frac{(n+1)^2 n!}{(n+1)!}$$

$$+ \frac{(n+1)^2 n!}{(n+2)!} + \frac{(n+1)^2 n!}{(n+3)!} + \cdots$$

B_n

$$A_n = \sum_{k=1}^{n} \frac{(n+1)^2 n!}{(k-1)!} \qquad B_n = \sum_{k>=2} \frac{(n+1)^2 n!}{(n+k)!}$$

$$(n+1)^2 n!e = \sum_{k=1}^{n} \frac{(n+1)^2 n!}{(k-1)!} + (n+1)^2 + (n+1) + \sum_{k>=2} \frac{(n+1)^2 n!}{(n+k)!}$$

$$(n+1)^2 n!e = A_n + (n+1)^2 + (n+1) + B_n \qquad (1)$$

$$(n+1)^2 n! e = A_n + (n+1)^2 + (n+1) + B_n \qquad (1)$$

Now if we consider A_n as an integer, we have to prove it.

$$A_n = \sum_{k=1}^{n} \frac{(n+1)^2 n!}{(k-1)!}$$

We will prove by induction:

Basis: $k=1, \quad A_1 = \frac{(n+1)^2 n!}{(1-1)!} = \frac{(n+1)^2 n!}{0!} = (n+1)^2 n! \in Z \quad (\because \mathrm{n} \in Z)$

Induction: Lets assume it holds for $k = n-1 \qquad \therefore A_{n-1} \in Z$

For $k=n, \quad A_n = A_{n-1} + \frac{(n+1)^2 n!}{(n-1)!} = A_{n-1} + (n+1)^2 n$

$$= \text{integer} + (\mathrm{n}+1)^2 n \qquad \text{(by induction hypothesis)}$$

$$= \text{integer} + \text{integer} \qquad \because \mathrm{n} \in Z$$

$$\therefore A_n \in Z$$

And $A_n = \sum_{k=1}^{n} \frac{(n+1)^2 n!}{(k-1)!} = n\sum_{k=1}^{n} \frac{(n+1)^2 (n-1)!}{(k-1)!} = nX_n$

The second portion $X_n = \sum_{k=1}^{n} \frac{(n+1)^2 (n-1)!}{(k-1)!}$ is an integer and we can prove it by a similar induction in the last page.

The basis : $k=1,\quad X_1 = \frac{(n+1)^2 (n-1)!}{(1-1)!} = \frac{(n+1)^2 (n-1)!}{0!} = (n+1)^2 (n-1)! \in Z \quad (\because \mathrm{n} \in Z)$

Taking $X_{n-1} \in Z$ as the induction hypothesis. We can show

For $k = n,\quad X_n = X_{n-1} + \frac{(n+1)^2 (n-1)!}{(n-1)!} = X_{n-1} + (n+1)^2$

$= \text{integer} + (\mathrm{n}+1)^2$ (by induction hypothesis)

$= \text{integer} + \text{integer} \quad \because \mathrm{n} \in Z$

$\therefore X_n \in Z$

So we can write A_n as follows

$\therefore A_n = nk \ \ (k \in Z)$

$$(n+1)^2 n! e = A_n + (n+1)^2 + (n+1) + B_n \qquad (1)$$

So we get 1) $\forall_{\mathrm{n}} A_n \in \mathrm{Z}$ and 2) $\forall_{\mathrm{n}} A_n = nk \quad , \mathrm{k} \in \mathrm{Z}$

Now we will focus on B_n

$$B_n = \sum_{k \geq 2} \frac{(n+1)^2 n!}{(n+k)!}$$

$$= \frac{(n+1)^2 n!}{(n+2)!} + \frac{(n+1)^2 n!}{(n+3)!} + \frac{(n+1)^2 n!}{(n+4)!} + \cdots$$

$$= \frac{(n+1)^2 n!}{(n+2)(n+1)n!} + \frac{(n+1)^2 n!}{(n+3)(n+2)(n+1)n!} + \frac{(n+1)^2 n!}{(n+4)(n+3)(n+2)(n+1)n!} + \cdots$$

$$= \frac{(n+1)}{(n+2)} + \frac{(n+1)}{(n+3)(n+2)} + \frac{(n+1)}{(n+4)(n+3)(n+2)} + \cdots$$

$$= \frac{(n+1)}{(n+2)} \left(1 + \frac{1}{(n+3)} + \frac{1}{(n+3)(n+4)} + \cdots \right) \qquad (2)$$

Now we know that $\because \frac{1}{x} < \frac{1}{(x-k)}$ Where k > 0 $\quad \therefore \frac{1}{n+4} < \frac{1}{n+3}$

Now from Eqn (2)

$$B_n = \frac{(n+1)}{(n+2)} \left(1 + \frac{1}{(n+3)} + \frac{1}{(n+3)(n+4)} + \cdots \right) < \frac{(n+1)}{(n+2)} \left(1 + \frac{1}{(n+3)} + \frac{1}{(n+3)(n+3)} + \cdots \right)$$

$$B_n < \frac{(n+1)}{(n+2)}(1+\frac{1}{(n+3)}+\frac{1}{(n+3)(n+3)}+\cdots) \quad (3)$$

$$B_n < \frac{(n+1)}{(n+2)}\sum_{i>=1}(\frac{1}{n+3})^{i-1}$$

The expression inside bracket is an infinite geometric progression

We know the formula

$$\sum_{i=1}^{\infty} a_i r^{i-1} = \frac{a_1}{1-r}$$

Here $a_1 = 1$ and $r = \frac{1}{n+3}$

So $\sum_{i>=1}(\frac{1}{n+3})^{i-1} = \frac{1}{1-\frac{1}{n+3}} = \frac{1}{\frac{n+3-1}{n+3}} = \frac{n+3}{n+2}$

Now plugging it into Eqn (3)

$$B_n < \frac{(n+1)(n+3)}{(n+2)(n+2)} = \frac{(n+1)(n+3)}{(n+2)^2} < 1$$

We also know the fact : $B_n = \sum_{k\geq 2}\frac{(n+1)^2 n!}{(n+k)!} \geq 0$ as $n >= 0$

So $\forall_{\text{n}} 0 \leq B_n < 1$

Now we have information that

1) $\forall_{\text{n}} A_n \in \mathrm{Z}$
2) $\forall_{\text{n}} A_n = nk \quad , \mathrm{k} \in \mathrm{Z}$
3) $\forall_{\text{n}} 0 \le B_n < 1$

$$\text{Now } (n+1)^2 n!e = A_n + (n+1)^2 + (n+1) + B_n \qquad (1)$$

We can take the floor of the both sides of Eqn (1)

$$\left\lfloor (n+1)^2 n!e \right\rfloor = \left\lfloor A_n + (n+1)^2 + (n+1) + B_n \right\rfloor$$

We know the formula $\lfloor x+n \rfloor = \lfloor x \rfloor + n, \quad$ integer n

$$\left\lfloor (n+1)^2 n!e \right\rfloor = A_n + (n+1)^2 + (n+1) + \lfloor B_n \rfloor$$

As $\forall_{\text{n}} 0 \le B_n < 1$

So $\lfloor B_n \rfloor = 0;$ as $\lfloor x \rfloor =$ The greatest integer less than or equal to x;

$$\therefore \left\lfloor (n+1)^2 n!e \right\rfloor = A_n + (n+1)^2 + (n+1)$$

Taking mod n on both sides

$$\lfloor (n+1)^2 n! e \rfloor \bmod n = (A_n + (n+1)^2 + (n+1)) \bmod n$$
$$= (nk + (n+1)^2 + (n+1)) \bmod n \qquad \because \forall_n A_n = nk \quad , k \in Z$$
$$= (nk + n^2 + 3n + 2) \bmod n$$
$$= (n(k+n+3)+2) \bmod n$$
$$= 2 \bmod n$$

So the simplified form of $\lfloor (n+1)^2 n! e \rfloor \bmod n$ is $2 \bmod n$