CSE547

Chapter 4, problem 15

Problem Statement

Does every prime occur as a factor of some Euclid number e_n ?

Needed Definitions

Euclid Number :

Definition :

Euclid numbers are integers of the form $En = p_n\# + 1$,

where p_n # is the **primorial of p_n (Definition follows)**which

is the *n*th prime.

Needed Definitions

They are named after the ancient Greek mathematician Euclid, who used them in his original proof that there are an infinite number of prime numbers.

Definition of Primorial: Let P denote prime numbers.

Assume that all P are put into an **increasing sequence** (P1≤P2 ≤ P3….. ≤ Pn) where Pn is the nth prime number.

(from wikipedia.org)

Given Pn we define **Primorial** of Pn (Pn#) as product of all prime numbers till Pn which is :

$$Pn\# = \Pi\ P_k \quad \text{(where k= 1 to n)}$$

Continued…

For $n \geq 2$, the **primorial** ($n\#$) is the product of all prime numbers less than or equal to n. For example, $7\# = 210$ is a primorial which is the product of the first four primes multiplied together ($2 \cdot 3 \cdot 5 \cdot 7$).

The simplest argument could be that to show that there is a prime number which is never the factor of any Euclid number.

If we consider any Euclid number, $P_n\#$ is always a multiple of 2. And Euclid number is 1 added to $P_n\#$.

Continued..

Every Euclid number is of the form

$= (2 * k) + 1$

where “k” is product of prime numbers<=n excluding 2.

So, it is very clear that there exists no Euclid number which is divisible by 2.

Answer

Hence, the answer is :

Every prime cannot occur as a factor of some Euclid number e_n.

Chapter 5,Problem No 16

- Evaluate the sum

$$\sum_{k} \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k$$

Where a, b, c are **NON NEGATIVE INTEGERS**

Continued…

The binomial coefficient $\binom{n}{k}$ can be expressed in terms of factorials as follows:

$$\binom{n}{k} = n!/(k!(n-k)!)$$

Continued…

Lets try to express each of the terms in the problem in factorials :

$$\binom{2a}{a+k} = \frac{(2a)!}{(2a-(a+k))!\ (a+k)!}$$

$$= \frac{(2a)!}{(a-k)!\ (a+k)!}$$

Continued…

Similarly,

$$\binom{2b}{b+k} = \frac{(2b)!}{(b-k)!\,(b+k)!}$$

$$\binom{2c}{c+k} = \frac{(2c)!}{(c-k)!\,(c+k)!}$$

Continued…

Therefore,

$$\sum_{k} \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$

$$= \sum_{k} \frac{(2a)!\,(2b)!\,(2c)!\,(-1)^k}{(a-k)!\,(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!}$$

Continued…

Multiplying numerator and denominator by

$(a+b)!\ \ (b+c)!\ \ (c+a)!$

We will therefore have,

$$= \sum_{k} \frac{[(2a)!\,(2b)!\,(2c)!]\ \ [(a+b)!\ (b+c)!\ (c+a)!\ ]\ \ (-1)^{k}}{[\ (a-k)!\,(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!\ ]\ \ [(a+b)!\ (b+c)!\ (c+a)!\ ]}$$

$$\frac{(2a)!\,(2b)!\,(2c)!}{(a+b)!\,(b+c)!\,(c+a)!} * \sum_k \frac{(a+b)!\,(b+c)!\,(c+a)!\,(-1)^k}{(a-k)!\,(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!}$$

Constant known form for

Lets try to get a this.

Continued…

Considering :

$$\sum_k \frac{(a+b)!\,(b+c)!\,(c+a)!}{(a-k)!\,(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!}$$

$$= \sum_k \frac{(a+b)!}{(a+k)!\,(b-k)!} \cdot \frac{(b+c)!}{(b+k)!\,(c-k)!} \cdot \frac{(c+a)!}{(c+k)!\,(a-k)!}$$

(Just interchanging the order of the terms in the denominator)

We know that,

$$\frac{(a+b)!}{(a+k)!\,(b-k)!} = \binom{a+b}{a+k}$$

Continued…

Similarly,

$$\frac{(b+c)!}{(b+k)!\,(c-k)!} = \binom{b+c}{b+k}$$

$$\frac{(c+a)!}{(c+k)!\,(a-k)!} = \binom{c+a}{c+k}$$

Continued…

Therefore,

$$\sum_{k} \frac{(a+b)!\,(b+c)!\,(c+a)!\,(-1)^k}{(a-k)!\,(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!}$$

$$= \sum_{k} \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} (-1)^k$$

which is a known form.

Using the equation given in Textbook Page No . 171,Eq. 5-29.

We have,

$$\sum \binom{a+b}{} \binom{b+c}{} \binom{c+a}{} (-1)^k = (a + b + c)!$$

Solution

Thus, the solution for the problem becomes :

$$\frac{(2a)!\,(2b)!\,(2c)!}{(a+b)!\,(b+c)!\,(c+a)!} \quad \frac{(a+b+c)!}{a!\;\;b!\;\;c!}$$