Chapter 5, Problem 7

Problem

- Is it true also when k < 0 ?

$$r^{\underline{k}}\left(r-\frac{1}{2}\right)^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$$

Observation1 (k > 0)

- Each term in the denominator of expanded r to the -k falling adds 2r with an even number, increasingly.

$$r^{\underline{-k}} = \frac{1}{(r+1)(r+2)\dots(r+k)}$$

$$= \frac{1}{(\frac{2r+2}{2})(\frac{2r+4}{2})\dots(\frac{2r+2k}{2})}$$

$$= \frac{2^k}{(2r+2)(2r+4)\dots(2r+2k)}$$

Observation2 (k > 0)

- Each term in the denominator of expanded (r−1/2) to the −k falling adds with an odd number, increasingly.

$$(r-\frac{1}{2})^{\underline{-k}} = \frac{1}{(r-\frac{1}{2}+1)(r-\frac{1}{2}+2)\ldots(r-\frac{1}{2}+k)}$$

$$= \frac{1}{(\frac{2r-1+2}{2})(\frac{2r-1+4}{2})\ldots(\frac{2r-1+2k}{2})}$$

$$= \frac{1}{(\frac{2r+1}{2})(\frac{2r+3}{2})\ldots(\frac{2r+2k-1}{2})}$$

$$= \frac{2^k}{(2r+1)(2r+3)\ldots(2r+2k-1)}$$

$$\frac{2^{k}}{(2r+2)(2r+4)\ldots(2r+2k)}$$

$$\times \frac{2^{k}}{(2r+1)(2r+3)\ldots(2r+2k-1)}$$

$$= \frac{2^{2k}}{(2r+1)(2r+2)(2r+3)\ldots(2r+2k-1)(2r+2k)}$$

- This result equals to 2r to the -2k falling times 2^{2k} .

$$\frac{2^{2k}}{(2r+1)(2r+2)(2r+3)\ldots(2r+2k-1)(2r+2k)}$$

- Thus,

$$r^{-k}(r-\frac{1}{2})^{-k} = (2r)^{-2k}2^{2k} = \frac{(2r)^{-2k}}{2^{-2k}}, k > 0$$

- As the problem ask for in case k < 0, we can set a k’ whose domain is negative integers; therefore we can replace −k with k’
- We can rewrite the formula as

$$r^{k'}(r-\frac{1}{2})^{k'} = \frac{(2r)^{2k'}}{2^{2k'}}, k' < 0$$

The Result

- As the domain of k’ is as same as the domain of k(that is less than zero) in the problem, we got the solution:

$$r^{\underline{k}}(r-\frac{1}{2})^{\underline{k}}=\frac{(2r)^{\underline{2k}}}{2^{2k}}$$

Is also true when $k < 0$.

Verifying the property

- In case k = -1,

$$r^{-1}(r-\frac{1}{2})^{-1} = \frac{4}{(2r+1)(2r+2)} = \frac{(2)^{-2}}{2^{-2}}$$

- In case k = -2,

$$r^{-2}(r-\frac{1}{2})^{-2} = \frac{16}{(2r+1)(2r+2)(2r+3)(2r+4)} = \frac{(2)^{-4}}{2^{-4}}$$