Problem 5.74

CSE547

Question

- This problems concerns a deviant version of Pascal's triangle in which the sides consist of numbers 1, 2, 3, 4, … instead of all 1's, although the interior numbers still satisfy the addition formula:

					1					
				2		2				
			3		4		3			
		4		7		7		4		
	5		11		14		11		5	
.		.		.		.		.		.

- If $\left(\binom{n}{k}\right)$ denotes the kth number in row n, for $1 \le k \le n$, we have $\left(\binom{n}{1}\right) = \left(\binom{n}{n}\right) = n$, and $\left(\binom{n}{k}\right) = \left(\binom{n-1}{k}\right) + \left(\binom{n-1}{k-1}\right)$ for $1 < k < n$. Express the quantity $\left(\binom{n}{k}\right)$ in closed form

Original Triangle

	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

Our Version

	$\left(\binom{n}{1}\right)$	$\left(\binom{n}{2}\right)$	$\left(\binom{n}{3}\right)$	$\left(\binom{n}{4}\right)$	$\left(\binom{n}{5}\right)$	$\left(\binom{n}{6}\right)$	$\left(\binom{n}{7}\right)$	$\left(\binom{n}{8}\right)$	$\left(\binom{n}{9}\right)$
1	1								
2	2	2							
3	3	4	3						
4	4	7	7	4					
5	5	11	14	11	5				
6	6	16	25	25	16	6			
7	7	22	41	50	41	22	7		
8	8	29	63	91	91	63	29	8	
9	9	37	92	154	182	154	92	37	9

Let's consider the two side-by-side

	$\left(\!\binom{n}{1}\!\right)$	$\left(\!\binom{n}{2}\!\right)$	$\left(\!\binom{n}{3}\!\right)$	$\left(\!\binom{n}{4}\!\right)$	$\left(\!\binom{n}{5}\!\right)$	$\left(\!\binom{n}{6}\!\right)$	$\left(\!\binom{n}{7}\!\right)$	$\left(\!\binom{n}{8}\!\right)$	$\left(\!\binom{n}{9}\!\right)$
1	1								
2	2	2							
3	3	4	3						
4	4	7	7	4					
5	5	11	14	11	5				
6	6	16	25	25	16	6			
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8	8	29	63	91	91	63	29	8	
9	9	37	92	154	182	154	92	37	9

	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
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6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

same

Let's consider the two side-by-side

	$\left(\binom{n}{1}\right)$	$\left(\binom{n}{2}\right)$	$\left(\binom{n}{3}\right)$	$\left(\binom{n}{4}\right)$	$\left(\binom{n}{5}\right)$	$\left(\binom{n}{6}\right)$	$\left(\binom{n}{7}\right)$	$\left(\binom{n}{8}\right)$	$\left(\binom{n}{9}\right)$
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8	8	29	63	91	91	63	29	8	
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Let’s consider the two side-by-side

	$\left(\!\binom{n}{1}\!\right)$	$\left(\!\binom{n}{2}\!\right)$	$\left(\!\binom{n}{3}\!\right)$	$\left(\!\binom{n}{4}\!\right)$	$\left(\!\binom{n}{5}\!\right)$	$\left(\!\binom{n}{6}\!\right)$	$\left(\!\binom{n}{7}\!\right)$	$\left(\!\binom{n}{8}\!\right)$	$\left(\!\binom{n}{9}\!\right)$
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After a little examination we observe the relationships above: every value in the left triangle = sum of two values in the right triangle of the same color

Found Formula

- Therefore the formula from the picture on the previous slide is given bellow:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n}{k-1} + \binom{n-1}{k}$$

Prove by Induction over n

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n}{k-1} + \binom{n-1}{k}$$

- Base Case: n = 1

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{1}{k} \qquad (1)$$

$$\binom{n}{k-1} + \binom{n-1}{k} = \binom{1}{k-1} + \binom{0}{k} \qquad (2)$$

- Two possibilities for k: k = 0 and k = 1
  - k = 0
    - (1) = 1; (2) = 0 + 1 = 1
  - k = 1
    - (1) = 1; (2) = 1 + 0 = 1

Prove by Induction over n

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n}{k-1} + \binom{n-1}{k}$$

- Inductive Case
  - First: expand based on the definition
  - Second: expand based on the formula

$$\left(\!\!\binom{n+1}{k}\!\!\right) = \left(\!\!\binom{n}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right) = \binom{n}{k-1} + \binom{n-1}{k} + \binom{n}{k-2} + \binom{n-1}{k-1}$$

$$\left(\!\!\binom{n+1}{k}\!\!\right) = \binom{n+1}{k-1} + \binom{n}{k} = \binom{n}{k-1} + \binom{n}{k-2} + \binom{n-1}{k} + \binom{n-1}{k-1}$$

- It is evident that the two are equal
- Therefore the formula is true!