CSE547 CH5.8

Question?

Evaluate $\sum_k \binom{n}{k} (-1)^k (1-\frac{k}{n})^n$ What is the approximate value of this sum, when n is very large? Hint: This sum is $\Delta^n f(0)$ for some function f.

Δ^n

- $\Delta^n f(0)$
- n is very large!

Formulas

Let’s remember the following formulas.

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$, integer n>0 (5.40)

Step 1

- Try to compare the given equation(1) and formula 5.40(2)
- (1) $\sum_{k}\binom{n}{k}(-1)^{k}(1-\frac{k}{n})^{n}$
- (2) $\Delta^{n} f(x)=\sum_{k}\binom{n}{k}(-1)^{n-k} f(x+k)$

Step2-1

Let's say $f(x) = (1 - \frac{k}{n})^n$, and apply $f(x)$ to formula 5.40. Then we have the following equations.

(1) $$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$ (5.40)

(2) $$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} (1 - \frac{x+k}{n})^n$$ (5.40)← $f(x)$

(3) $$\Delta^n f(0) = \sum_k \binom{n}{k} (-1)^{n-k} (1 - \frac{k}{n})^n$$ x ← 0

Step2-2

(4) $$\Delta^n f(0) = \sum_k \binom{n}{k} (-1)^{(k-n)} (1-\frac{k}{n})^n$$

(5) $$\Delta^n f(0) = (-1)^{-n} \sum_k \binom{n}{k} (-1)^k (1-\frac{k}{n})^n$$

Finally, we induce the given equation,

$\sum_k \binom{n}{k} (-1)^k (1-\frac{k}{n})^n$ on the right side of (5)

$$\therefore \sum_k \binom{n}{k} (-1)^k (1-\frac{k}{n})^n = (-1)^{-n} \Delta^n f(0)$$

Step 3

Let's remember the nth difference of a Newton series

$$\Delta^n f(0) = \begin{cases} c_n & \text{if } n < d; \\ 0 & \text{if } n > d; \end{cases}$$

Here, $c_n = n!a_n$

Now, we can change

$$\sum_k \binom{n}{k} (-1)^k (1 - \frac{k}{n})^n = (-1)^{-n} \Delta^n f(0)$$

into

$$\therefore \sum_k \binom{n}{k} (-1)^k (1 - \frac{k}{n})^n = (-1)^{-n} n! a_n$$

Step 4

- Let's think about the Binomial Expansion to get a_n

$$(1-\frac{k}{n})^n = 1 + (-1)\binom{n}{1}\left(\frac{x}{n}\right)^1 + \ldots\ldots + (-1)^n\binom{n}{n}\left(\frac{x}{n}\right)^n$$

$$\therefore a_n = (-1)^n\binom{n}{n}\left(\frac{x}{n}\right)^n$$

Step 5

- Now, let's think about the given condition that n is large.

$$\sum_k \binom{n}{k}(-1)^k (1-\frac{k}{n})^n = (-1)^{-n} n! a_n$$

$$\sum_k \binom{n}{k}(-1)^k (1-\frac{k}{n})^n = (-1)^{-n} n! (-1)^{-n} \left(\frac{1}{n}\right)^n = (-1)^{-2n} n! \left(\frac{1}{n}\right)^n$$

$$= (-1)^{-2n} \frac{n!}{n^n} = 0$$

$$(\because \lim_{n->\infty} \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \ldots\ldots\ldots \cdot \frac{n-1}{n} \cdot \frac{n}{n} = 0)$$