CHAPTER 5

pp 153 - 243

BINOMIAL COEFFICIENTS

DEF 1

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 2\cdot 1} = \frac{n^{\underline{k}}}{k!}$$

$$\boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}}$$

FOR $k \geq 0$, $n, k \in \mathbb{N}$, $k \leq n$

COMBINATORIAL INTERPRETATION

$\binom{n}{k}$ - "n choose k"

$\binom{n}{k}$ denotes a NUMBER of ways to choose K-ELEMENT SUBSET from an n-element SET.

PROOF of COMBINATORIAL STATEMENT

$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ is a number of ways to choose k-elem SUBSETS from n-elem SET.

STEP 1

Find the number of k-element 1-1 sequences formed out of n-element SET

REMARK 1 - all sequences of lenght k from n-elem set are all function

$f : \{1, .. k\} \to \{a_1 .. a_n\}$

and there are n^k of them.

We need 1-1 sequences only.

REMARK 2

DEF PERMUTATION of $A = \{a_1, ... a_n\}$, $|A| = n$

is any FUNCTION $f : A \xrightarrow[\text{onto}]{1-1} A$

We prove:

If $|A| = n$, then # of PERMUTATIONS of A is $n!$ $n > 0$

PROOF of Remark 2

$$\boxed{\text{If } |A| = n \text{, then there are } n! \text{ functions } f.\quad f : A \xrightarrow[\text{onto}]{1-1} A \quad, n \geq 1}$$

By induction over $|A| = n$

$n = 1$ $A = \{a\}$ and there is one functio[n]

$f : \{a\} \xrightarrow[\text{onto}]{1-1} \{a\}$, $f(a) = a$ as $1! = 1$ yes

Assume that $B \subsetneq A$ (any subset of A) and $|B| = n-1$, then there is $(n-1)!$ functions that map $f : B \xrightarrow[\text{onto}]{1-1} B$

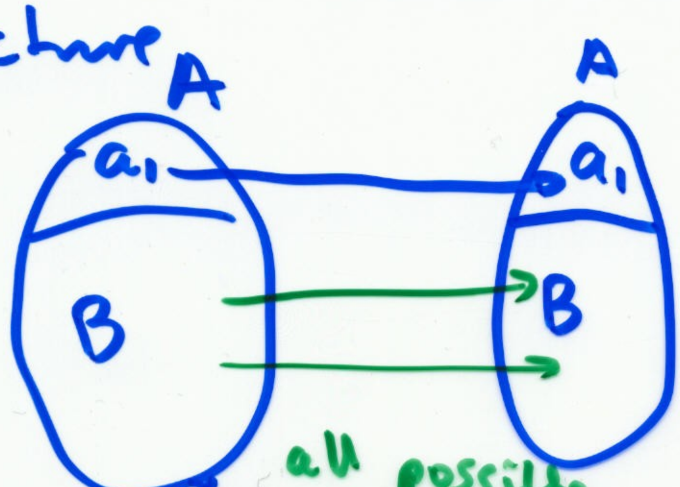
We count $f : A \xrightarrow[\text{onto}]{1-1} A$ as follows.

Assume $A = \{a_1, a_2 \ldots a_n\}$

GROUP 1

$\boxed{f(a_1) = a_1}$ Fixed a_1 By ind assumption there are $(n-1)!$ such functions

Picture



all possible $g : B \xrightarrow[\text{onto}]{1-1} B$

$|B| = n-1$

$f : A \xrightarrow[\text{onto}]{1-1} A$

$\boxed{f(a_1) = a_1, \; f(b) = g(b)}$ for $b \in B$

$g(b) = b'$

$\boxed{\text{GROUP 2}}$

$\boxed{f(a_2) = a_2}$ $\boxed{\text{Fix } a_2}$ + $\boxed{f(b) = g(b)}$

for $b \in B = \{a_1, a_3 \ldots a_n\}$ $|B| = n-1$

There are $(n-1)!$ functions in G2

$\boxed{\text{Group } i}$ $i \geq 2$

$\boxed{f(a_i) = a_i}$ + $\boxed{f(b) = g(b)}$

for $b \in A - \{a_i\} = B$ $g : B \xrightarrow[\text{onto}]{1\text{-}1} B$

There are $(n-1)!$ functions

There are n - groups ; all together there are

$n \cdot (n-1)!$ functions $f : A \xrightarrow[\text{onto}]{1\text{-}1}$

$\boxed{n! = n(n-1)!}$ end of proof of REMARK 2

In particular $A = \{a_1, \ldots a_k\}$

$\boxed{\text{There are } k! \text{ 1-1 sequences of length } k \text{ out of } A.}$

BACK TO STEP ONE

Let $|A| = n$ be an n-element set

We construct all 1-1 k-element sequences out of elements of A as follows: sequence 1-1, all $i \neq j$ $b_i \neq b_j$

$$\boxed{b_1 b_2 \dots b_K}$$

(1) b_1 — there are n choises, $b_1 \in A$

(2) elem $b_1 b_2$ — there are $n-1$ choises of $b_2 \neq b_1$

(3) elem $b_1 b_2 b_3$ — there are $n-2$ choises for $b_3 \neq b_2 \neq b_1$, $b_i \in A$

Induction (really)

$b_1 b_2 \; b_K$ — there are $(n-k+1)$ choises for $b_K \neq b_{k-1} \dots \neq b_1$

All together we have

$$\boxed{n(n+1)\dots(n-k+1)}$$

possible 1-1 sequences $b_1 b_2 \dots b_K$

$\boxed{\text{STEP 2}}$ $\binom{n}{k}$ represents how many are there k-element $\boxed{\text{SUBSETS}}$

We know that there are $n(n+1)\ldots(n-k+1)$ sequences 1-1

$$\boxed{b_1 b_2 \ldots b_k} \longleftrightarrow \{b_1 b_2 \ldots b_k\}$$

1-1 sequence ? subset

{ **SETS**: $\{b_1 b_2 b_3 \ldots b_k\} = \{b_2 b_1 b_3 \ldots b_k\}$ etc

different SEQUENCES represent the same SET

How many of all possible "representations" – as many as PERMUTATIONS i.e $k!$

$$\boxed{\binom{n}{k}} = \frac{\#\text{sequences}}{k!} = \boxed{\frac{n(n+1)\ldots(n-k+1)}{k!}}$$

END

GENERALIZATION

We proved (COMBINATORIAL INT)

$$\boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}}$$

$\boxed{k, n \in N \quad k \leq n}$

is a number of k-elem SUBSETS of an n-elem SET.

$\boxed{\begin{array}{l} 0! = 1 \\ x^{\underline{0}} = 1 \end{array}}$ $\binom{0}{0} = \frac{0^{\underline{0}}}{1} = 1$

$\boxed{\begin{array}{l} n^{\underline{n}} = n! \\ 1^{\overline{n}} = n! \end{array}}$ $\binom{n}{n} = \frac{n^{\underline{n}}}{n!} = \frac{n!}{n!} =$

We had

$f : R \to R$

$\boxed{f(x) = x^{\underline{k}} = x(x-1)\ldots(x-k+1)}$

or Really

$f : R \times N \to R$

$f(x, k) = x^{\underline{k}}$

$f : R \to R$ for g.v. $k \in Z$

We DEFINE $\boxed{f : R \times Z \to R}$

or even

$f : C \times Z \to R$

C - complex numbers

$$\boxed{f(x, k) = \begin{cases} \frac{x^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}}$$

$f(x, k) = \binom{x}{k}$ = g(x) - for any $k \in Z$

EXAMPLE

$$\boxed{\binom{x}{k} = \frac{x^{\underline{k}}}{k!}} \qquad k \geq 0$$

$$= 0 \qquad k < 0$$

$$\binom{-1}{3} = \frac{(-1)^{\underline{3}}}{3!} = \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} = -1$$

$$\boxed{x^{\underline{k}} = x(x-1)\cdots(x-k+1)}$$

$$\boxed{\binom{n}{n} = 1 \quad n \in \mathbb{N}}$$

$$\binom{-1}{-1} = 0 \qquad (k < 0) \quad !$$

$\binom{1}{1} = 1$ in general $\boxed{\binom{n}{n} = 1}$ $n \geq 0$

only when $n \geq 0$

$\boxed{\binom{n}{n} = 0 \quad n < 0}$

$$\boxed{\binom{\sqrt{2}}{3} = \frac{(\sqrt{2})^{\underline{3}}}{3!}} = \boxed{\frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{6}}$$

$$(\sqrt{2})^{\underline{3}} = \sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)$$

NO COMBINATORIAL INTERPRETATION!

We defined $f: R \times Z \to R$ ⑧

$f(x, k) = \binom{x}{k}$

$$\binom{x}{k} = \begin{cases} \dfrac{x^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where $x \in R$, $k \in Z$

$f: R \times Z \to R$

$f(x, k) = \binom{x}{k}$

$k! = 1^{\overline{k}}$ $k \in Z$

use x to denote that UPPER LIMIT is a real number or

$k! = k^{\underline{k}}$

We use $\to$ $n \in N$, $k \in N$

$f: N \times N \to R$

$$\boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}}$$

For COMBINATORIAL interpretation

BOOK USES $r \in R$ instead $x \in R$

$$\binom{r}{k} = \begin{cases} \dfrac{r^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$f(r, k) = \binom{r}{k}$

$f: R \times Z \to R$

SAME FUNCTION

$$\boxed{\binom{x}{k} = \frac{x^{\underline{k}}}{k!} \quad k \geq 0, \quad \binom{x}{k} = 0 \quad k < 0 \qquad k \in \mathbb{Z},\ x \in \mathbb{R}}$$

REMINDER

1. $\binom{n}{n} = 1 \qquad n \in \mathbb{N}$

2. $\binom{n}{n} = 0 \qquad n < 0$

3. $\binom{n}{k} = 0$ for $k > n$, $k \geq 0$

we have restriction $k \geq 0$ in $\boxed{\binom{x}{k} = \frac{x^{\underline{k}}}{k!}}$ i.e $\binom{x}{k} = 0 \quad k < 0$

Let's look at

SYMMETRY IDENTITY

CONSIDER $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$0 \leq k \leq n$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-(n-k))!\,(n-k)!} = \binom{n}{n-k}$$

↑ proof

SYMMETRY PROPERTY

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

for $k, n \in \mathbb{N}$, $k \leq n$

COMBINATORIAL INTERPRETATION

$\binom{n}{k}$ k chosen elements out of n

$\binom{n}{n-k}$ n-k "unchosen" elements out of n.

SYMMETRY PROPERTY C.D

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

Proved $k, n \in$

$k \leq n$

restriction

$n \in N$

CASE $k < 0$

$$\binom{n}{k} = 0 \quad , \quad \binom{n}{n-k} = \binom{n}{s} = 0$$

$s > n$

CASE $k > n$

$$\binom{n}{k} = 0 \quad , \quad \binom{n}{n-k} = \binom{n}{s} = 0$$

$s < 0$

So we have

SYMMETRY GENERAL

$$\boxed{\binom{n}{k} = \binom{n}{n-k} \qquad n \in N \ , \ k \in Z}$$

$f : N \times Z \to N$

why we can't have $n \in Z$?

Check $\binom{n}{k} = \binom{n}{n-k}$ for $n = -1$

$k \geq 0$ (L)=

$$\boxed{\binom{-1}{k}} = \frac{(-1)^{\underline{k}}}{k!} = \frac{(-1)(-2)\ldots(-\cancel{1}-k+\cancel{1})}{k!} \leftarrow k \text{ factors}$$

$$= \frac{(-1)^{k}\,k!}{k!} = \boxed{(-1)^{k}} \qquad x^{\underline{k}} = x(x-1)\ldots(x-k+1)$$

(R)=

$$\boxed{\binom{-1}{-1-k}} = 0 \quad \text{all } k \geq 0$$

By induction.

$$\binom{-1}{k} \neq \binom{-1}{-1-k} \quad \text{all } k \geq 0.$$

ABSORPTION IDENITY

$$\boxed{\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}}$$

$k \neq 0$

$x \in \mathbb{R}$

$k \in \mathbb{Z} \setminus \{0\}$

PROOF OF ABSORPTION IDENTITY

① $$\boxed{\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}} \qquad x\in R,\ k\neq 0,\ k\in Z$$

iff

② $$\boxed{k\binom{x}{k} = x\binom{x-1}{k-1}} \qquad x\in R,\ k\in Z$$

Proof:

① Observe $$\boxed{x^{\underline{k}} = x(x-1)^{\underline{k-1}}}$$

GENERAL $$\boxed{x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}}$$

L $$x^{\underline{k}} = x(x-1)\cdots(x-k+1)$$

R $$x(x-1)^{\underline{k-1}} = x(x-1)(x-2)\cdots((x-1)-(k-1)+1)$$

$$= x(x-1)\cdots(x-1-k+1+1)$$

$$= x(x-1)\cdots(x-k+1) = x^{\underline{k}}$$

Evaluate:

$$\binom{x}{k} = \frac{x(x-1)^{\underline{k-1}}}{k\cdot(k-1)!} = \frac{x}{k}\binom{x-1}{k-1}$$

end

ABSORPTION IDENTITY

PROVED

2 $$\boxed{k\binom{x}{k} = x\binom{x-1}{k-1}}$$ $x \in R$, $k \in Z$

TO BE PROVED:

3 $$\boxed{(x-k)\binom{x}{k} = x\binom{x-1}{k}}$$

BIG DOMAIN

$$\frac{x \in R}{k \in Z}$$

$$L \quad = \quad P$$

$N \subseteq R$

Proof

1 Prove ~~Assume~~ the case $\boxed{x \in N}$, $k \in Z$

$$L = (x-k)\binom{x}{k} \overset{\text{Symmetry}}{=} (x-k)\binom{x}{x-k}$$

use 2 for $k := x-k$

1 $\binom{n}{k} = \binom{n}{n-k}$ SYMMETRY

2
$$= x\binom{x-1}{x-k-1}$$

$$= x\binom{\overset{n}{x-1}}{\underset{n-k}{(x-1)-k}} \overset{\text{Symmetry}}{=} x\binom{x-1}{k} = P$$

$\binom{n}{k}$

1/2 end

We proved identity 3 for a case $\boxed{x \in \mathbb{N},\ k \in \mathbb{Z}}$.

Now we are going to show that this rezults $\boxed{\text{EXTENDS}}$ to $\boxed{x \in \mathbb{R}}$ $k \in \mathbb{Z}$

We do it by what is called

$\boxed{\text{A POLYNOMIAL argument.}}$

OBSERVE:

$$\boxed{(x-1)\binom{x}{k} = x\binom{x-1}{k}} \qquad \begin{matrix} x \in \mathbb{R} \\ k \in \mathbb{Z} \end{matrix}$$

is EQUALITY OF TWO POLYNOMIALS OF DEGREE $(k+1)$

$$\boxed{L(x)} = (x-1)\binom{x}{k} = \boxed{a_{k+1} x^{k+1} + \dots + a_0}$$

$$\boxed{P(x)} = x\binom{x-1}{k} = \boxed{b_{k+1} x^{k+1} + \dots + b_0}$$

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k(k-1)!}$$

polynomial of degree k over $\mathbb{Z}$.

POLYNOMIALS THEOREMS

THM 1

Let $w(x) = a_n x^n + \dots + a_0$ $\quad a_n \neq 0$

be a polynomial of the degree n

The equation

$$w(x) = 0$$

has at most n solutions; i.e

$$|\{x : w(x) = 0\}| \leq n.$$

THM 2

If $w(x) = a_n x^n + \dots a_0$ $\quad a_n \neq 0$

is such that

$$|\{x : w(x) = 0\}| > n$$

then $\boxed{w(x) = 0 \quad \text{for all } x \in \mathbb{R}}$

BACK TO OUR IDENTITY

$L(x) = (x-1)\binom{x}{k} = a_{k+1}x^{k+1} \cdots + a_0$

$P(x) = x\binom{x-1}{k} = b_{k+1}x^{k+1} + \cdots b_0$

3 $\boxed{L(x) = P(x)}$ for all $x \in R$, $k \in K$

$\updownarrow$

GOAL: all $x \in R$, $k \in K$

$\underbrace{L(x) - P(x)}_{w(x)} = 0$

We have a polynomial $w(x) = L(x) - P(x)$ of degree $k+1$

$\boxed{|\{x : L(x) - P(x) = 0| = x_0 > k+1}$ $(\cancel{x_0 \in R})$ $\to$ we proved f $x \in \mathbb{N}$

because we proved that $L(x) = P(x)$ for all $x \in N$.

by thm 2 $w(x) = L(x) - P(x) = 0$ for all $x \in R$.

HENCE

$L(x) = P(x)$, for all $x \in R$, $k \in K$.

$$\boxed{(x-k)\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k}}$$ ③ Independent proof Ba

$$x\binom{x}{k} - k\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k}$$

② $\Vert$

$$x\binom{x}{k} - x\binom{x-1}{k-1} \stackrel{?}{=} x\binom{x-1}{k}$$

$$x\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k} + x\binom{x-1}{k-1}$$

$$x\binom{x}{k} \stackrel{?}{=} x\left(\underbrace{\binom{x-1}{k} + \binom{x-1}{k-1}}\right)$$

$$\underline{\binom{x-1}{k}} \stackrel{?}{=} \frac{(x-1)(x-2)\ldots\ldots(x-1-k+1)}{k!}$$

$x-\cancel{1}-k+\cancel{1}+1$ $$= \frac{(x-1)(x-2)\ldots\ldots(x-k)}{k\,(k-1)!}$$

$$\binom{x-1}{k-1} = \frac{(x-1)\ldots\ldots(x-k+1)}{(k-1)!}$$

$$= \frac{(x-1)^{\underline{k-1}}}{(k-1)!}\left(1 + \frac{x-k}{k}\right)$$

$$\left(1 + \frac{x}{k} - 1\right)$$

$$x\binom{x}{k} \stackrel{?}{=} \binom{x-1}{k-1}\cdot\frac{x}{k}\cdot x$$

$x \neq 0$

$$\binom{x}{k} \stackrel{?}{=} \binom{x-1}{k-1}\frac{x}{k}$$

$$k\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k-1}$$

yes was proven

+ check case

$x = 0$

SYMMETRY PROPERTY

(1) $$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$ $\boxed{n \in N}$ $k \in Z$

ABSORBTION IDENTITY

(2) $$\boxed{\binom{n}{k} = \frac{x}{k}\binom{x-1}{k-1}}$$ $\boxed{x \in R}$ $k \in Z - \{0\}$

(3) $$\boxed{k\binom{x}{k} = x\binom{x-1}{k-1}}$$ $\boxed{x \in R}$, $k \in Z$

PROVE FIRST FOR $x \in N$, and then extend it via POLYNOMIAL ARGUMENT to $x \in R$

(4) $$\boxed{\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}}$$ $x \in R$ $k \in Z$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

$k \geq 0$, $k \leq n$
$n, k \in N$

when $n \in R$ we put

0 $k < 0$

NEXT IDENTITY

(4) $$\boxed{\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}}$$ $x \in R$, $k \in \mathbb{Z}$

CASE 1 Assume $\boxed{x \in N}$, i.e $n \in N$

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$$

and we have a COMBINATORIAL INTERPRETATION:

$\binom{n}{k}$ = # of k-element subsets chosen from the n-elem. set

If we have n-eggs with exactly one bad egg, we have

$\binom{n-1}{k}$ selections that have good eggs

$\binom{n-1}{k-1}$ of them contain the BAD egg because they have k-1 of n-1 good eggs.

PROVED

SO

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}} \qquad n \in N,\ k \in Z$$

all eggs — only good — contain bad egg

\+ POLYNOMIAL ARGUMENT

$L(x) = \binom{x}{k}$ polynomial of the degree k $\quad x \in R$

$P(x) = \binom{x-1}{k} + \binom{x-1}{k-1}$ polynomial of the degree k

we proved

polyn on degree k

$L(x) - P(x) = 0$ for all $x \in N$

by thm 2

$L(x) - P(x) = 0$ for all $x \in R$, $k \in$

$$\boxed{L(x) = P(x) \quad \text{for all } x \in R,\ k \in K}$$

PROOF 2

Use identities 3 + 2

$x \in \mathbb{R}$
$k \in \mathbb{Z}$

We know

3 $$(x-k)\binom{x}{k} = x\binom{x-1}{k},$$ 2 $$k\binom{x}{k} = x\binom{x-1}{k-1}$$

$\boxed{\text{ADD THEM}}$: 3 + 2

$$(x-k)\binom{x}{k} + k\binom{x}{k} = x\binom{x-1}{k} + x\binom{x-1}{k-1}$$

$$\binom{x}{k}(x-k+k) = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right)$$

we get:

$$\binom{x}{k}x = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right) \quad *$$

$x \neq 0$

$$\boxed{\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}}$$

check $x=0$

Check CASE $\boxed{x=0}$

$$\boxed{\binom{0}{k} = \binom{-1}{k} + \binom{-1}{k-1}} \;?$$

① $\boxed{k<0}$ use: $\binom{x}{k}=0$ for all $k<$

L=R

$0 = 0 + 0$

$\binom{-1}{k} = \frac{(-1)^k}{k!}$ for $k \geq 0$, $=0$ for $k<$

$\binom{-1}{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$ for $k \geq 1$, $=0$ for $k<1$

$\boxed{x^0 = 1}$

② $\boxed{k=0}$

$\binom{0}{0} \stackrel{?}{=} \binom{-1}{0} + \binom{-1}{-1}$

$1 = 1 + 0$

$\binom{-1}{0} = \frac{(-1)^0}{1} = 1$

$\binom{n}{k} = 0$ $k>n$

$\binom{n}{n} = 0$, $n<0$

(3) $\boxed{k>0\ ;\ k\geq 1}$

$$\boxed{\binom{0}{k} = \binom{-1}{k} + \binom{-1}{k-1}}$$

$k-1<$

$k<1$

$$\binom{-1}{k} = \frac{(-1)^{\underline{k}}}{k!} \qquad \binom{-1}{k-1} = \frac{(-1)^{\underline{k-1}}}{(k-1)!}$$

L =

$$\binom{0}{k} = \frac{0^{\underline{k}}}{k!} = 0$$

USE

$$\boxed{x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}}$$

Evaluate

P =

$$\binom{-1}{k} + \binom{-1}{k-1} =$$

$$= \frac{(-1)^{\underline{k}}}{k!} + \frac{(-1)^{\underline{k-1}}}{(k-1)!}$$

$$\boxed{(-1)^{\underline{k}}} = (-1)^{\overset{m\ +\ n}{\underline{(k-1)+1}}}$$

$$= (-1)^{\underline{k-1}}(-1-k+1)^{\underline{1}}$$

$$= \boxed{(-1)^{\underline{k-1}}(-k)}$$

$$= \frac{(-1)^{\underline{k-1}}(-\cancel{k})}{(k-1)!\,\cancel{k}} + \frac{(-1)^{\underline{k-1}}}{(k-1)!}$$

$$= \frac{(-1)^{\underline{k-1}}}{(k-1)!}((-1)+1) = 0$$

$$\boxed{L = P.}$$

Proof 3

$$\boxed{\binom{x-1}{k} + \binom{x-1}{k-1}} \overset{\text{Def}}{=} \frac{(x-1)^{\underline{k}}}{k!} + \frac{(x-1)^{\underline{k-1}}}{(k-1)!}$$

$$= \frac{(x-1)^{\underline{k-1}}(x-k)}{k!} + \frac{(x-1)^{\underline{k-1}} \cdot k}{k!}$$

$$= \frac{(x-1)^{\underline{k-1}}}{k!}\left(x-k+k\right)$$

$$= \frac{x \cdot (x-1)^{\underline{k-1}}}{k!} = \frac{x^{\underline{k}}}{k!} = \boxed{\binom{x}{k}}$$

$k \geq 0$, $k-1 \geq 0$, $k \geq 1$

OBSERVE

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \underbrace{\binom{2}{0}}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \underbrace{\binom{1}{-1}}_{=0} \quad \text{STOP}$$

GENERAL (proof by induction)

$$\boxed{\sum_{0 \le k \le n} \binom{x+k}{k} = \binom{x}{0} + \binom{x+1}{1} + \dots \binom{x+n}{n} = \binom{x+n+1}{n} \qquad n \in \mathbb{Z}}$$

when $k < 0$ factors are 0. GUESS

We guessed

(1) $$\boxed{\sum_{k\le n}\binom{x+k}{k}=\binom{x+n+1}{n}}$$ $n\in\mathbb{Z}$, $k\in\mathbb{Z}$, $x\in\mathbb{R}$

We use now formula

$$\boxed{\binom{x}{k}=\binom{x-1}{k}+\binom{x-1}{k-1}}$$

to unfold (differently than before)

$$\underset{m+1}{\overset{n+1}{\binom{5}{3}}}=\underbrace{\binom{4}{3}}_{\text{unfold}}+\binom{4}{2}=\underbrace{\binom{3}{3}}_{\text{unfold}}+\binom{3}{2}+\binom{4}{2}$$

$$=\binom{2}{3}+\binom{2}{2}+\binom{3}{2}+\binom{4}{2}$$

$$=\binom{1}{3}+\binom{1}{2}+\binom{2}{2}+\binom{3}{2}+\binom{4}{2}$$

$$=\binom{0}{3}+\binom{0}{2}+\binom{1}{2}+\binom{2}{2}+\binom{3}{2}+\underset{m}{\overset{n}{\binom{4}{2}}}$$

and we guess a new formula:

$$\sum\binom{k}{m} \qquad m=2$$

We guess

m CONSTANT

(2) $$\boxed{\sum_{0\le k\le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots \binom{n}{m} = \binom{n+1}{m+1}}$$

Need $k \geq 0$

for $k<0$ same

Both guesses (1) and (2) can $\binom{k}{m} \neq 0$
be proved by math induction
We will prove (2) by induction
and then we will prove (2) from (2)
combinatorial interpretation of (2):

If we want to choose $(m+1)$ tickets
from a set of $(n+1)$ tickets
numbered $0, 1, \ldots n$
there are $\binom{k}{m}$ ways of doing this
when the largest ticket selection
number is k

$$\binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots \binom{n}{m} = \binom{n+1}{m+1}$$

$k=0$ $k=1$

Proof by induction of (2)

$$\boxed{\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}}$$

$\boxed{n, k, \cancel{m} \in N}$

Induction over $n \in N$

$n=0$ $\quad \sum_{k=0}^{0}\binom{k}{m} = \binom{0}{m} \overset{?}{=} \binom{1}{m+1}$ $\qquad \binom{0}{0}=1$

⊗

$\binom{1}{m+1} = \binom{0}{m+1} + \binom{0}{m} = 0$, $\quad \binom{0}{m} = 0 \quad m>0$

IND $\quad m>0$

Assume

$$\boxed{S_{n-1} = \binom{n}{m+1}}$$

Prove

$S_n = \sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$

$k \in Z$

$$S_n = S_{n-1} + \binom{n}{m}$$

$$\boxed{\binom{n+1}{m+1} \overset{?}{=} \overset{\text{Ind}}{\binom{n}{m+1}} + \binom{n}{m}}$$

use:

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\binom{n+1}{m+1} = \binom{n}{m+1} + \binom{n}{m}$$

$\boxed{\begin{array}{l} x = n+1 \\ k = m+1 \end{array}}$

$n = x-1$

$k-1 = m$

yes.

⊗ $m=0 \quad \binom{1}{1} = 1 = \binom{0}{0} = 1$

PROOF of ① from ②

$m \in N$

$n \in Z$

① $$\boxed{\sum_{k \le n} \binom{m+k}{k} = \binom{m+n+1}{n}}$$

② $$\boxed{\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}}$$

$n, m \in N$

USE $$\binom{n}{k} = \binom{n}{n-k} \quad (*)$$

$n \in N$

$k \in Z$

Evaluate for $m+k \geq 0$, $k \geq -m$

out $k < -m$

but terms are 0

$$\sum_{k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{k}$$

$$\overset{(*)}{=} \sum_{-m \le k \le n} \binom{m+k}{m} = \sum_{k=0}^{n+m} \binom{k}{m}$$

Replace

$k \to k-m$

$-m \le k \le n$

$\downarrow$

$-m \le k-m \le n$

$0 \le k \le n+m$

$$\overset{②}{=} \binom{m+n+1}{m+1} \overset{(*)}{=} \binom{m+n+1}{n}$$

$$\left(\underbrace{m+n+1}_{n} - \underbrace{(m+1)}_{k} = n\right)$$

Consider (8) SPECIAL CASES

$$\boxed{\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}}$$

PROVED

$m, n \in N$

Consider $\boxed{m = 1}$

$$\boxed{\binom{n+1}{2}} = \sum_{k=0}^{n}\binom{k}{1} = \binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \dots \binom{n}{1}$$

$$= 0 + 1 + 2 + \dots n = \boxed{\frac{n(n+1)}{2}}$$

General case $\binom{k}{m}$:

$$\sum_{k=0}^{n}\binom{k}{m} = \sum_{k=0}^{n}\frac{k^{\underline{m}}}{m!} = \frac{1}{m!}\sum_{k=0}^{n}k^{\underline{m}}$$

$\parallel$

$\binom{n+1}{m+1}$

$$= \binom{n+1}{m+1}$$

i.e.

$$\boxed{\sum_{k=0}^{n}k^{\underline{m}} = m!\binom{n+1}{m+1}}$$

FORMULA

[illegible]

c.d

$$\boxed{\sum_{k=0}^{n} k^{\underline{m}} = m!\binom{n+1}{m+1}} = \qquad \boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}}$$

$$= \frac{m!\,(n+1)^{\underline{m+1}}}{(m+1)!}$$

$$= \frac{(n+1)^{\underline{m+1}}}{m+1}$$

SPECIAL SUM.

$$\boxed{\sum_{k=0}^{n} k^{\underline{m}} = \frac{(n+1)^{\underline{m+1}}}{(m+1)}}$$

$n, m \in \mathbb{N}$

YOU CAN GET IT BY INTEGRATION (short)

Look back at

$$\boxed{\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}}$$

– re-write it as

$x = x+1$

$$\binom{x+1}{m} = \binom{x}{m} + \binom{x}{m-1} \quad (*)$$

and evaluate

$$\Delta\left(\binom{x}{m}\right) = \binom{x+1}{m} - \binom{x}{m} = \binom{x}{m-1}$$

DEFINITION

from $(*)$

$$\Delta\left(\binom{x}{m}\right) = \binom{x}{m-1}$$ $x \in \mathbb{R},\ m \in \mathbb{Z}$

and "integral"

$$\sum \binom{x}{m} \delta x = \binom{x}{m+1} + C$$

We get a SHORT PROOF of (2):

$$\sum_{x=0}^{n} \binom{x}{k} = \binom{x}{k+1}\Bigg|_{0}^{n+1} = \binom{n+1}{k+1} - \binom{0}{k+1}$$

$$= \binom{n+1}{k+1}$$

used

$$\sum_{k=a}^{b-1} g(k) = \sum_{a}^{b} g(x)\,\delta x$$