CHAPTER 2 :

SUMS

INTRODUCTION :

1. sequences and
2. sums of sequences

1. **DEFINITION** (of a SEQUENCE of elements of a SET A)

A SEQUENCE of elements of a set A is ANY FUNCTION f

$$f : N \to A$$

, N set of NATURAL NUMBERS

$N = \{0, 1, 2 \dots\}$

Any

$$\boxed{f(n) = a_n}$$

is called $\boxed{n\text{-th TERM}}$ of a sequence f.

NOTATION

$f = \{a_n\}$ or $\{a_n\}_{n \in N}$, $\{a_n\}$

$\boxed{\text{EXAMPLE}}$ of a sequence of REAL numbers

N-SET OF INDEXES

$f: \mathbb{N} \to \mathbb{R}$

$$\boxed{\begin{aligned} f(n) &= n + \sqrt{n} \\ a_n &= n + \sqrt{n} \end{aligned}}$$

or $\boxed{\text{SHORTHAND}}$

$a_n = n + \sqrt{n}$

$a_0 = 0$, $a_1 = 1+1 = 2$, $a_2 = 2 + \sqrt{2}$ ....

$0, 2, 2+\sqrt{2}, 3+\sqrt{3}, \ldots\ldots n+\sqrt{n}, \ldots$

$\boxed{\text{OBSERVATION 1}}$

Sequence is ALWAYS INFINITE (countably infinite) as by definition, domain of f is a set $\mathbb{N}$ of nat. numbers.

$A \sim B$ iff $|A| = |B|$

$\boxed{\text{OBSERVATION 2}}$

$\mathbb{N} \sim \mathbb{N} - \{0\} = \mathbb{N}^+$

$\mathbb{N} \sim \mathbb{N} - K$, K any finite subset of $\mathbb{N}$

$\mathbb{N} \sim \mathbb{Z}$, $\mathbb{N} \sim \text{ODD}$, $\mathbb{N} \sim \text{EVEN}$

In general: a set T is called $\boxed{\text{COUNTABLY INFINITE}}$ iff $|T| = |N|$ i.e there is a function

$$f : N \xrightarrow[\text{onto}]{1\text{-}1} T \qquad (T \sim N)$$

$\boxed{\text{OBSERVATIONS}}$

We can choose a a SET of INDEXES of a sequence any COUNTABLY infinite set T, not only N.

$\boxed{\text{IN OUR BOOK}}$: $T = N - \{0\} = N^+$

i.e sequences "start" with $n=1$

$$a_1, a_2, a_3 \ldots \qquad \{a_n\}_{n \in N^+}$$

GENERAL FORM of a SEQUENCE.

$\boxed{\text{FINITE SEQUENCE}}$: any f

$f : K \to A$, where $|K| = n$ $n \in N$

or K - finite subset of N.

FINITE SEQUENCE (2)

$f : \{1, \dots n\} \to A \quad , n \in N$

$f(n) = a_n$

$\{a_k\}_{k=1..n}$

$\boxed{a_1 \; a_2 \dots a_n}$ $\quad k = 1, 2 \dots n$

EMPTY SEQUENE : case $n = 0$

e $\qquad f(\phi) = e$

DOMAIN of the SEQUENCE

$\boxed{f : T \to A}$

DOMAIN

FORMULA:

$$\boxed{a_n = \frac{n}{(n-2)(n-5)}}$$

DOMAIN of f is

$T = N - \{2, 5\}$ $\quad$ i.e

$T = \{-1, -2, 3, 4\}$

$$\boxed{\begin{array}{c} f : N - \{2,5\} \to R \\ f(n) = a_n \end{array}}$$

FINITE SEQUENCE

FINITE.

$\boxed{T \subseteq Z}$

② $\boxed{\text{SUMS}}$ of elements of a sequence of RATIONAL #.

In CHAPTER 2 we consider only $\boxed{\text{FINITE SUMS}}$ of consequitive elements of a sequences $\{a_n\}$ of RATIONAL NUMBER

DEFINITION

Given a sequence $\boxed{f : N^+ \to R}$ of RATIONAL NUMBERS

$$\boxed{f(n) = a_n}$$

$$a_1 \quad a_2 \quad a_3 \ldots \quad a_n \ldots$$

We write

$$\boxed{\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n}$$

$$\sum_{k=1}^{n} a_k = \sum_{1 \le k \le n} a_k = \sum_{k \in \{1, \ldots n\}} a_k = \sum_{K} a_k$$

for $K = \{1, \ldots n\}$

Given a sequence of numbers:

$\boxed{\begin{array}{c} f: N^+ \to R \\ f(n) = a_n \end{array}}$ or $\boxed{\begin{array}{c} a_1\ a_2\ \dots a_n \dots \\ a_k \in R \end{array}}$

FULL DEFINITION SHORTHAND

we sometimes need to evaluate a SUM of some sub-sequence of $\{a_n\}$ only.

For example: sum-up only each second term of $\{a_n\}$ (i.e $n \in$ EVEN)

We write it two ways:

1. $$\boxed{\sum_{\substack{1 \le K \le 2n \\ K \in \text{EVEN}}} a_K = a_2 + a_4 + \dots + a_{2n}}$$

P(K) summation property

2. $$\boxed{\sum_{K=1}^{n} a_{2K} = a_2 + a_4 + \dots + a_{2n}}$$

subsequence property

WE USE NOTATION

$$\boxed{\sum_{P(k)} a_k = \sum_{k \in K} a_k}$$

for

$$\boxed{K = \{ n \in N : \ P(n) \}}$$

$P(n)$ - is a certain formula (predicate) defining our restriction on n.

We assume

① $\boxed{K \text{ is defined}}$; i.e
$P(n) = T$ or F is decidable

② $\boxed{K \text{ is FINITE}}$
(we consider only finite sums for a moment)

EXAMPLE 1

Let $\boxed{P(n) = (1 \le n < 100) \wedge (n \in ODD)}$

$P(n)$ is a formula (open) defining all ODD numbers between 1 and 99 (included)

$$\boxed{K} = \{n \in N : P(n)\}$$

$$= \{1, 3, 5 \ldots, 99\}$$

$$2n+1 = 99$$

$$2n = 98$$

$$n = 49$$

$$= \{n \in N : 1 \le n \le 99\}$$

$$= \boxed{\{1, 3, \ldots (2n+1)\}} \text{ for } \boxed{0 \le n \le 49}$$

$$\boxed{\sum_{P(n)} a_n = \sum_{k \in K} a_k}$$

$$= \boxed{\sum_{n=0}^{49} a_{(2n+1)} = a_1 + a_3 \ldots + a_{99}}$$

EXAMPLE 2

Let

$$\boxed{P(n) = (1 \leq n < 100)}$$

$P(n)$ is a predicate defining all $n \in \mathbb{N}$ between 1 and 99 included.

$$K = \{1, 2 .. \; 99\}$$

In this case

$$\boxed{\sum_{P(n)} a_n = \sum_{k \in K} a_k = \sum_{k=1}^{99} a_k}$$

$$= a_1 + a_2 + a_3 + .. + a_{99}$$

EXAMPLE 3

Let $\boxed{a_n = (2n+1)^2}$

$\boxed{P(n) = (1 \leq n < 100)}$

$f(n) = a_n$
DOMAIN f
$= \mathbb{N}$

Evaluate : $\displaystyle\sum_{P(n)} a_n$ $\quad K = \{1, \dots 99\}$

$$\boxed{\sum_{P(n)} (2n+1)^2} = \sum_{k=1}^{99} (2n+1)^2 = \boxed{3^2 + 5^2 + \cdots (2 \cdot 99 + 1)^2}$$

BOOK NOTATION

p.24-25

(from Kenneth Iverson's programming language APL)

$$\boxed{[P(x)] = \begin{cases} 1 & P(x) \text{ true} \\ 0 & P(x) \text{ false} \end{cases}}$$

Example

$$[p \text{ prime}] = \begin{cases} 1 & p \text{ is prime} \\ 0 & p \text{ is not prime} \end{cases}$$

We write

$$\boxed{\sum_{P(k)} a_k = \sum_{k} a_k [P(k)]} = \sum_{k \in K} a_k$$

$$K = \{k \in N : P(k)\}$$

we can put K out.

$\boxed{\text{BOOK NOTATION c.d}}$ 1

Example

$$\boxed{\sum_{p} [p \text{ prime}][p \le n] \, \frac{1}{p}}$$

Property

$$a_p = \frac{1}{p}$$

$P(x)$ is $P_1(x) \cap P_2(x)$

for $P_1(x)$: x is prime

$P_2(x)$: $x \le n$, for $n \in N$

$P(x)$ says : x is prime and $x \le n$

$\sum_p$ means : we sum over all p that are PRIME and $p \le n$

CASE $n = 0$ $P(x)$ is FALSE

PRIMES are natural numbers ≥ 2 etc..

BOOK NOTATION C.d ② 53a

Book uses notation

$p \le N$ for $p \le n$ $n \in N$

where N denotes a natural number. IT is NOT CORRECT

N ALWAYS denotes a set of Natural numbers

I will use

$p \le n$ $n \in N$

When you read the book now and later, pay attention

it happens often

$n \le K$ for all $n \in K$,

means that $n \le K$, for some K

Usually $\boxed{\text{CAPITAL LETTERS DENOTE SETS.}}$

BOOK NOTATION c.d 3

1. Authors never define a sequence

$$\{a_n\}$$

for $\sum a_k$

2. They say often

"a_k" is defined / not defined for all set of INTEGERS"

IT MEANS they admitt FINITE sequences

$f: K \to A$ $\qquad f(k) = a_k$

for K finite subset of $\mathbb{Z}$

BOOK NOTATION c.d (4)

$$\boxed{\sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_{k} [P(k)]\, a_k}$$

where

$$\boxed{K = \{k \in \mathbb{Z} : P(k)\} \quad \text{and } K \text{ is FINITE.}}$$

YOU CAN PUT

we admit this case

$K = \{k \in R : P(k)\}$ or K - finite

$$[P(k)] = \begin{cases} 1 & P(k) \text{ TRUE} \\ 0 & P(k) \text{ FALSE} \end{cases}$$

$$\boxed{K = \{k \in \mathbb{N} : P(k)\} \quad \text{and } K \text{ is FINITE}}$$

This is usual case.

EXAMPLE 4

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} \rightarrow \text{changes limits}$$

SUMS AND RECURRENCES

Observation, for any $n \in N$

$$\boxed{\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}}$$

CASE $n=0$

$$\sum_{k=1}^{1} a_k = a_1 \qquad \sum_{k=1}^{0} a_k \stackrel{\text{def}}{=} 0$$

DEFINE:

$$\boxed{\sum_{k=a}^{b} a_k = 0 \quad \text{when } b<a}$$

In General when sum is UNDEFINED we put it to 0

SUMS AND RECURRENCES

Observation: for all $n \in \mathbb{N}$

$$\boxed{\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_n}$$

$n=0$

$$\sum_{k=0}^{0} a_k = a_0$$

$$\sum_{k=0}^{-1} a_k = 0 \qquad a_n = a_0$$

We get $\quad a_0 = 0 + a_0 = a_0$

$n=1$

$$\sum_{k=0}^{1} a_k = a_0 + a_1$$

$$\sum_{k=0}^{0} a_k = a_0 \qquad a_n = a_1$$

$$a_0 + a_1 = a_0 + a_1$$

We know

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_k + a_n \quad (*)$$

Denote

$$\boxed{S_n = \sum_{k=0}^{n} a_k}$$

SUM FUNCTION

$$\boxed{\begin{gathered} S : N \to R \\ S(n) = S_n = \sum_{k=0}^{n} a_k \end{gathered}}$$

We re-write $(*)$ as

Recursive form (RECURRENCE)

$$\boxed{\begin{aligned} S_0 &= a_0 \\ S_n &= S_{n-1} + a_n \end{aligned}}$$

for $n > 0$

We can use techniques from CHAPTER 1 to evaluate (if possible) CLOSED formulas for SUMS.

EXAMPLE

Given a sequence

$$f : N \to R \qquad f(n) = a_n$$

$$\boxed{a_n = a + b \cdot n}$$

PARAMETERS

$a, b \in R$

CONSTANTS

Find a CLOSED FORMULA for

$$\boxed{S(n) = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a + b \cdot k)}$$

The Reccurence form of $S(n)$

RS

$$\boxed{\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + \underbrace{(a + bn)}_{a_n} \end{aligned}}$$

$S : N \to R$

$S(n) = S_n$

We want to find a CF for this reccurence formula.

WE CONSIDER A MORE GENERAL CASE

$$\boxed{\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + (\beta + \gamma n) \end{aligned}} \quad (R)$$

(RS) is a case of (R) for

$$\boxed{\begin{aligned} \alpha &= a \\ \beta &= a \\ \gamma &= b \end{aligned}}$$

and look for (CF)

$\boxed{\text{STEP 1}}$: Evaluate few terms

$R_0 = \alpha$

$R_1 = \alpha + \beta + \gamma$

$R_2 = \alpha + \beta + \gamma + (\beta + 2\gamma) = \alpha + 2\beta + 3\gamma$

$R_3 = \alpha + 2\beta + 3\gamma + (\beta + 3\gamma) = \alpha + 3\beta + 6\gamma$

$\vdots$

$\boxed{\text{STEP 2}}$: Obsevation:

IN GENERAL

CF

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma}$$

GOAL: FIND $A(n)$, $B(n)$, $C(n)$

and this proves R = CF

METHOD: Repertoir method

STEP 1 SET $R_n = 1$, all $n \in N$

i.e $R(n) = 1 = R_n$ is a constant function (if possible)

use R $\boxed{R_0 = \alpha, \quad R_n = R_{n-1} + (\beta + \gamma \cdot n)}$

$R_0 = 1$ gives $\alpha = 1$

$R_n = R_{n-1} + (\beta + \gamma \cdot n)$ gives

$1 = 1 + (\beta + \gamma \cdot n)$ for all $n \in N$

Evaluate:

We get

$$\boxed{0 = \beta + \gamma \cdot n \qquad \text{for all } n \in \mathbb{N}}$$

This is possible only when

$$\beta = \gamma = 0$$

We obtained (for $R_n = 1$, all n)

$$\boxed{\alpha = 1, \quad \beta = 0, \quad \gamma = 0}$$

And our closed formula

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma}$$

becomes

$$\boxed{R_n = A(n)\alpha = A(n) = 1, \text{ all } n \in \mathbb{N}}$$

We proved

FACT 1 $\boxed{A(n) = 1, \text{ for all } n \in \mathbb{N}}$

$\boxed{\text{STEP 2}}$

We set R_n ~~CONSTANT~~

$$\boxed{R_n = n, \quad \text{all } n}$$

and find α, β, γ if exist.

R $\boxed{R_0 = \alpha, \quad R_n = R_{n-1} + (\beta + \gamma n)}$

$R_0 = \alpha = 0$ gives $\boxed{\alpha = 0}$

$R_n = R_{n-1} + (\beta + \gamma n)$ becomes

$n = (n-1) + \beta + \gamma n$ for all n

$$\boxed{1 = \beta + \gamma n \quad \text{for all } n}$$

possible only when

$$\boxed{\beta = 1, \quad \gamma = 0}$$

We evaluate CF for

$$\boxed{\alpha = 0, \quad \beta = 1, \quad \gamma = 0}$$

CF

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma}$$

becomes for $\alpha = 0,\ \beta = 1,\ \gamma = 0$ and $R_n = n$, all n

$$n = A(n)\cdot 0 + B(n)\cdot 1 + C(n)\cdot 0$$

FACT 2

$$\boxed{B(n) = n \quad \text{for all } n \in N}$$

STEP 3

We set

$$\boxed{R_n = n^2}, \text{ all } n \in N$$

and find α, β, γ, if exist.

$A(n)$ is a function $\quad A : N \to R$

$B(n) \quad B \quad\quad A(n) \in R$

STEP 3 c.d

RF

$$\boxed{R_0 = \alpha, \quad R_n = R_{n-1} + (\beta + \gamma n) \qquad R_n = n^2, \text{ all } n}$$

$R_0 = \alpha = 0^2$ $\qquad \boxed{\alpha = 0}$

$n^2 = (n-1)^2 + \beta + \gamma n$, all $n \in N$

$n^2 = n^2 - 2n + 1 + \beta + \gamma n$, all $n \in N$

$0 = -2n + 1 + \beta + \gamma n$, all $n \in N$

$\boxed{0 = (1+\beta) + n(\gamma - 2)} \Leftrightarrow$ all $n \in N$

iff

$1 + \beta = 0$ $\qquad \gamma - 2 = 0$

$\boxed{\beta = -1}$ $\qquad \boxed{\gamma = 2}$

We get:

$\boxed{\alpha = 0, \ \beta = -1, \ \gamma = 2}$ and calculate CF

CF

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \quad \text{for } R_n = n^2 \text{ all } n \in \mathbb{N}, \quad \alpha = 0,\ \beta = -1,\ \gamma = 2}$$

$$n^2 = -B(n) + 2C(n) \quad \text{all } n \in \mathbb{N}$$

We know (FACT2) that

$$B(n) = n, \text{ all } n \in \mathbb{N}$$

$$n^2 = -n + 2C(n) \quad \text{all } n \in \mathbb{N}$$

FACT3

$$\boxed{\frac{n^2+n}{2} = C(n)}$$

Put (FACT 1) + 2 + 3 + CF

$\boxed{A(n) = 1}$

CF

$$\boxed{R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma}$$

GO BACK to

$$\boxed{S_n = \sum_{k=0}^{n} (a + bk)}$$

$$S_n = R_n \quad \text{for } \alpha = a,\ \beta = a,\ \gamma = b$$

$$R_n = \alpha + n\beta + \left(\frac{n^2+n}{2}\right)\gamma$$

$$S_n = a + na + \left(\frac{n^2+n}{2}\right) b$$

$$\boxed{S_n = (n+1)a + \frac{n(n+1)}{2} b}$$

and we evaluated

$$\boxed{\sum_{k=0}^{n} (a + bk) = (n+1)a + \frac{n(n+1)}{2} b}$$

Of course we can do it by a simpler method

$$\boxed{\sum_{k=0}^{n}(a+bk) =}$$

We use Properties of summation to be listed next

P1

$$= \sum_{k=0}^{n} a + \sum_{k=0}^{n} b\cdot k$$

ev

P2

$$= (n+1)a + b\sum_{k=0}^{n} k$$

eval

$$= (n+1) + \frac{n(n+1)}{2}\, b$$

$a_n = a$, all n

eval $\sum_{k=0}^{n} a_n = \sum_{k=0}^{n} a = \underbrace{a + \cdots + a}_{n+1} = (n+1)a$

SUMMATION PROPERTIES

(p. 30)

P1 $$\sum_{k\in K} c\cdot a_k = c\sum_{k\in K} a_k$$ DISTRIBUTIVE LAW

P2 $$\sum_{k\in K}(a_k+b_k) = \sum_{k\in K} a_k + \sum_{k\in K} b_k$$ ASSOCIATIVE LAW

P3 $$\sum_{k\in K} a_k = \sum_{\pi(k)\in K} a_{\pi(k)}$$ COMMUTATIVE LAW

$\pi(k)$ - any PERMUTATION of elements of K (real numbers)

$\boxed{K \text{ - FINITE subset of INTEGERS}}$

SUM OF GEOMETRIC SEQUENCE

$\boxed{\text{GEOMETRIC SUM}}$

DEF

$f: \mathbb{N} \to \mathbb{R}$ is $\boxed{\text{geometric}}$ iff

$f(n) = a_n$ $\quad \boxed{\forall_{n\in\mathbb{N}} \left(\frac{a_{n+1}}{a_n} = q \right)}$

constant

We prove

FACT

$$\forall_{n\in\mathbb{N}} \boxed{a_n = a_0 q^n}$$ for geometric sequence

all $n \in \mathbb{N}$

GEOMETRIC SUM = SUM OF A GEOMETRIC SEQUENCE

$S: \mathbb{N} \to \mathbb{R}$

$S(n) = S_n$

$$\boxed{S_n = \sum_{k=0}^{n} a_0 q^k = \frac{a_0(1-q^{n+1})}{1-q}}$$

$$S_n = a_0 + a_0 q + \dots + a_0 q^n$$

$$q \cdot S_n = a_0 q + a_0 q^2 + \dots + a_0 q^n + a_0 q^{n+1}$$

$$S_n(1-q) = a_0 - a_0 q^{n+1}$$

GEOMETRIC SUM

$$\boxed{S_n = \sum_{k=0}^{n} a_0 q^n = \frac{a_0(q^{n+1}-1)}{(q-1)}}$$

Example

$$\boxed{S_n = \sum_{k=0}^{n} 2^{-k} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k}$$

$a_0 = 1 \qquad q = \frac{1}{2}$

$$S_n = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{-1/2} \; (-2) = 2 - \left(\frac{1}{2}\right)^{n}$$

$$\boxed{S_n = \sum_{k=1}^{n} 2^{-k} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k} \qquad S_n = \frac{a_1(q^{n+1}-1)}{(q-1)}$$

PREVIOUS S_n minus 1!

$a_1 = \frac{1}{2} \qquad q = 1/2$

OR $$S_{n-1} = \frac{\frac{1}{2}\left(\left(\frac{1}{2}\right)^{n} - 1\right)}{-1/2} = 1 - \left(\frac{1}{2}\right)^{n}$$

RECCURENCE → SUM → CLOSED FORMULA

(R) (CF)

GOAL 69

(R) $\boxed{\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}}$ (T)

$T : N \to R$

Divide by 2^n

(R') $T_0/2^0 = 0$

$$T_n/2^n = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$$

DENOTE $\boxed{S_n = T_n/2^n}$

(RS) $\boxed{\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + \frac{1}{2^n} \end{aligned}}$

IT Means

(S) : $N \to R$

$T = S$

$\forall n \; T(n) = S(n)$

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

We know (as S_n is GEOMETRIC)

$$\boxed{S_n = 1 - \frac{1}{2^n}}$$

We use

$$S_n = T_n / 2^n$$

$$\boxed{T_n = 2^n S_n}$$

to get a CLOSED FORMULA CF for T_n

$$T_n = 2^n \left(1 - \frac{1}{2^n}\right) = 2^n - 1$$

CF $\boxed{T_n = 2^n - 1}$