$\boxed{\text{MULTIPLE SUMS}}$

$\boxed{\text{EXAMPLE 1}}$ DUBLESUM (TWO factors)

$$\sum_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} a_i b_j = a_1 b_1 + a_1 b_2 + a_1 b_3$$
$$+ a_2 b_1 + a_2 b_2 + a_2 b_3$$
$$+ a_3 b_1 + a_3 b_2 + a_3 b_3$$

or ANY PERMUTATION in ORDER.

$\boxed{\text{QUESTION:}}$

CAN WE EXPRESS IT IN TERMS

OF $\sum_i a_i$, $\sum_j b_j$?

and HOW ?

① FIRST: DEFINITION

$$\sum_{\substack{1\le i\le 3\\ 1\le j\le 3}} a_i b_j = \sum_{1\le i\le 3}\Big(\sum_{1\le j\le 3} a_i b_j\Big)$$

$$= \sum_{1\le j\le 3}\Big(\sum_{1\le i\le 3} a_i b_j\Big)$$

DEF GENERAL.

$i\in I,\ j\in J$

$$\boxed{\sum_{i,j} a_i b_j \overset{\text{DEF}}{=} \sum_i \sum_j a_i b_j = \sum_j \sum_i a_i b_j}$$

Where we write

$$\boxed{\sum_{i,j} a_i b_j \quad \text{for} \quad \sum_{\substack{i\in I\\ j\in J}} a_i b_j}$$

and $\boxed{\sum_i \sum_j a_i b_j}$ for $\boxed{\sum_i \Big(\sum_j a_i b_j\Big)}$

② RELATIONSHIP (Example 1)

$$\sum_{\substack{i,j \\ i,j \in \{1,2,3\}}} a_i b_j = a_1 b_1 + a_1 b_2 + a_1 b_3 + a_2 b_1 + a_2 b_2 + a_2 b_3 + a_3 b_1 + a_3 b_2 + a_3 b_3$$

$$= a_1 (b_1 + b_2 + b_3) + a_2 (b_1 + b_2 + b_3) + a_3 (b_1 + b_2 + b_3)$$

pull out common factor

$$= (b_1 + b_2 + b_3)(a_1 + a_2 + a_3)$$

$$= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$$

We proved

$$\boxed{\sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j = \left(\sum_{i=1}^{3} a_i\right)\left(\sum_{j=1}^{3} b_j\right) = \left(\sum_{j=1}^{3} b_i\right)\left(\sum_{i=1}^{3} a_i\right) = \sum_{i,j} a_i b_j}$$

QUESTION :

Can we GENERALIZE ?

It means can we prove

General DISTRIBUTIVE LAW

$$\boxed{\sum_{\substack{i\in I\\ j\in J}} a_i b_j = \left(\sum_{i\in I} a_i\right)\left(\sum_{j\in J} b_j\right)}$$

We need to bring in our general definitions

$$\boxed{\sum_{i\in I} a_i} = \sum_{P(i)} a_i = \boxed{\sum_{i\in I} a_i [P(i)]}$$

$$\boxed{I = \{ i : P(i) \}}$$

predicate defining I set of indices

$$[P(i)] = \begin{cases} 1 & P(i) \text{ true} \\ 0 & P(i) \text{ false.} \end{cases}$$

CHARACTERISTIC FUNCTION OF $P(i)$

$$\boxed{\sum_{j\in J} b_j = \sum_{Q(j)} b_j = \sum_{j} b_j [Q(j)]}$$

$J = \{j : Q(j)\}$

$Q(j)$ is a predicate defining set J of indices.

We re-write the GENERAL DISTRIBUTIVE LAW

$$\sum_{\substack{i\in I \\ j\in J}} a_i b_j = \left(\sum_i a_i [P(i)]\right)\left(\sum_j b_j [Q(j)]\right)$$

QUESTION: HOW TO RELATE LEFT SIDE TO RIGHT SIDE?

① we know $\boxed{\sum_{P(k)} a_k}$: ①

what is $\sum_{\substack{i\in I \\ j\in I}} a_i b_j$ ②, $\sum_{P(i,j)} a_{ij}$

⓪ what is a_{ij} ?

$\boxed{\text{Abstract (DESCRETE) MATH.}}$

a_n is n-th term of a sequence

$f: N \to R$ $\qquad f(n) = a_n$

one index

We define $\boxed{\text{2-INDICES SEQUENCE}}$

$\boxed{\text{is a function } f: N \times N \to R}$

(A - in general!)

We write

$$\boxed{f(i,j) = a_{ij}}$$

and call a_{ij} (ij) term of the sequence (two-indices seq)

We can have sequences of many indices:

a_{ijk}, $a_{i_1 i_2 i_3 i_4}$ ... $a_{i_1 \dots i_k}$

IN GENERAL:

MULTIPLE INDICES SEQUENCES

DEF

Any function $k \geq 1$

$$f : N^k \to A \qquad N^k = \underbrace{N \times N \ldots \times N}_{k}$$

$$f(n_1, \ldots n_k) = a_{n_1 \ldots n_k}$$

is called a k-indeces sequence (of elements of a set $A \neq \emptyset$)

FINITE SEQUENCE

$$f : K \to A$$

K is a finite set $K \subseteq N^k$, $k \geq 1$

$$N^k = \underbrace{N \times N \ldots \times N}_{k}$$

Many argument PREDICATES

$P(x_1)$, $P(x_1, x_2)$ ... $P(x_1, \dots x_k)$

FOR FINITE SEQUENCES THE SET K (domain) is OFTEN DEFINED by a PREDICATE

i.e

$\mathbf{x} \in \mathrm{dom}\, f$

$$K = \{x : P(x)\}$$

$\leftrightarrow$ defines LIMITS of SUMMATION

① $\mathrm{Dom} f = N$ — ONE index seq

$$K = \{i \in N : P(i)\}$$

(sometimes $i \in Z$.)

② $\mathrm{Dom} f = N \times N$

$$K = \{(i, j) : P(i, j)\}$$

TWO arg Predicate
TWO INDICES sequence

③ $\mathrm{Dom} f = N \times N \times N$

$$K = \{(i, j, k) : P(i, j, k)\}$$

THREE

OBSERVATION

$$\boxed{\begin{aligned} K &= \{(n_1 \dots n_k) : P(n_1 \dots n_k)\} \\ &= K_1 \times K_2 \times \dots \times K_k \subseteq N^k \end{aligned}}$$

$n_1 \in K_1,\ n_2 \in K_2 \dots,\ n_k \in K_k$

If we have PREDICATES $P_1(x) \dots P_k(x)$ that define respectively sets $K_1, K_2 \dots K_m$

WE WRITE MULTIPLE SUM

DEF

$$\sum_{P(n_1 \dots n_k)} a_{n_1 \dots n_k} = \sum_{P_1(n_1)} \sum_{P_2(n_2)} \dots \sum_{P_k(n_k)} a_{n_1 \dots n_k}$$

OR

$$\sum_{(n_1 \dots n_k) \in K} a_{n_1 \dots n_k} = \sum_{n_1 \in K_1} \sum_{n_2 \in K_2} \dots \sum a_{n_1 \dots n_k}$$

IN GENERAL - K-INDICES SEQUENCE

$f : N^k \to R$ (or $A \neq \emptyset$)

$f(n_1 \dots n_k) = a_{n_1 \dots n_k}$

$$\boxed{K = \{(n_1 \dots n_k) : P(n_1 \dots n_k)\}}$$

where $\boxed{P(n_1 \dots n_k)}$ is a PREDICATE ~~DOMAIN~~

defining a DOMAIN of

the FINITE SEQUENCE = Finite SUMS!

$K \subseteq N \times \dots \times N$

$$\boxed{\sum_K a_{n_1 \dots n_k} = \sum_{P(n_1 \dots n_k)} a_{n_1 \dots n_k} = \sum_{n_1 \dots n_k} a_{n_1 \dots n_k} [P(n_1 \dots n_k)]}$$

OUR MULTIPLE SUMS and PREDICATES DEFINING their DOMAIN (limits of summation)

Example 1

TAKE $\boxed{c_{ij} = a_i b_j}$ $\sum_{i,j} c_{ij} =$? ?

$$= \sum_{i,j \in \{1,2,3\}} a_i b_j = \sum_{\substack{1 \le i \le 3 \\ 1 \le j \le 3}} a_i b_j = \sum_{1 \le i,j \le 3} a_i b_j$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j = \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j$$

we proved: $= \left(\sum_{i=1}^{3} a_i\right)\left(\sum_{j=1}^{3} b_j\right)$

$$= \left(\sum_{j=1}^{3} b_j\right)\left(\sum_{i=1}^{3} a_i\right)$$

GOAL: Prove GENERAL DISTRIBUTION LAW

$$\boxed{\sum_{\substack{i \in I \\ j \in J}} a_i b_j = \left(\sum_{i \in I} a_i\right)\left(\sum_{j \in J} b_j\right)}$$

IN OUR EXAMPLE 1

$$P(i,j) = (1 \le i \le 3) \wedge (1 \le j \le 3)$$

AND

$$= (1 \le i,j \le 3)$$

$$P_1(i) = (1 \le i \le 3)$$

$$P_2(j) = (1 \le j \le 3)$$

$$\boxed{P(i\,j) = P_1(i) \wedge P_2(j)}$$

DEF

$$\boxed{\sum_{P(i,j)} a_{ij} = \sum_{P_1(i)} \sum_{P_2(j)} a_{ij}}$$

$$\sum_{1 \le i,j \le 3} a_{ij} = \sum_i \sum_j a_{ij} \, [P(i,j)]$$

$$= \sum_i \sum_j a_{ij} \, [P_1(i) \wedge P_2(j)]$$

DUBLE

$$\boxed{P(i\,j) = P_1(i) \wedge P_2(j)}$$

SUMMATION split of SUMMATION

BUT in our EXAMPLE 1 we had

IN GENERAL

$$\boxed{\sum_{P(i,j)} a_i b_j}$$

$$\boxed{\sum_{P(i,j)} a_{ij}}$$ Know this

How it connects with a_{ij} ?

DOMAIN of a_{ij} is $K_1 \times K_2$

We define

SPECIFIC DOUBLE IND. SEQUENCE

$$\boxed{a_{ij} = a_i \cdot b_j}$$

DOMAIN of a_i is K_1

DOMAIN of b_j is K_2

where

$a_{ij} = f(i,j)$ for $\boxed{f : N \times N \to R}$ general

FOR SUM LIMITS

$$f : \underbrace{K_1 \times K_2}_{K} \to R$$

$a_i = g(i)$

for $g : K_1 \to R$

$b_j = h(j)$

$h : K_2 \to R$

$$\boxed{f(i,j) = g(i) \cdot h(j)}$$

all $(i,j) \in K_1 \times K_2$

We want to prove (generalization of EXAMPLE 1

$$\boxed{\sum_{P(i,j)} a_i b_j = \Big(\sum_{P_1(i)} a_i\Big)\Big(\sum_{P_2(j)} b_j\Big)}$$

where $\boxed{P(i,j) = P_1(i) \wedge P_2(j)}$ $\in I \times J$

In PARTICULAR we re-write it

$$\boxed{\sum_{\substack{i \in I \\ j \in J}} a_i b_j = \Big(\sum_{i \in I} a_j\Big)\Big(\sum_{j \in J} b_j\Big)}$$

$P(i,j)$ $P_1(i)$ $P_2(j)$

$$\boxed{\sum_{1 \le i,j \le 3} a_i b_j = \Big(\sum_{i=1}^{3} a_i\Big)\Big(\sum_{j=1}^{3} b_j\Big)}$$

Let $\boxed{P(i,j) = P_1(i) \cap P_2(j)}$

$$\boxed{\sum_{P(i,j)} a_i b_j} = \sum_{i,j} a_i b_j \, [P(i,j)]$$

just proved

$$(*) \quad = \sum_{i,j} a_i b_j \, [P_1(i)][P_2(j)]$$

Def

$$= \sum_i \left(\sum_j a_i b_j \, [P_1(i)][P_2(j)] \right)$$

constant w.r.t. j

Pull out a_i

CONST w.r.t. i

$$= \sum_i \left(a_i [P_1(i)] \cdot \sum_j b_j [P_2(j)] \right)$$

$$= \left(\sum_j b_j [P_2(j)] \right) \cdot \left(\sum_i a_i [P_1(i)] \right)$$

$$= \boxed{\left(\sum_{P_1(i)} a_i \right) \left(\sum_{P_2(j)} b_j \right)}$$

$(*)$ We use $\boxed{[P_1(i) \cap P_2(j)] = [P_1(i)][P_2(j)]}$

True for any CHARACTERISTIC FUNCTION

$$\boxed{\sum_{\substack{P_1(i)\\P_2(j)}} a_i b_j = \left(\sum_{P_1(i)} a_i\right)\left(\sum_{P_2(j)} b_j\right)}$$

OR

GENERAL DISTRIBUTIVE LAW

$$\boxed{\sum_{\substack{i\in I\\j\in J}} a_i b_j = \left(\sum_{i\in I} a_i\right)\left(\sum_{j\in J} b_j\right)}$$

$\{a_i\}$

EXAMPLE (of application)

CONSIDER the ARRAY $(n\times n)$

$$\begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & a_{23} & \\ \vdots & & & \\ a_na_1 & a_na_2 & \cdots & a_n\cdot a_n \end{bmatrix}$$

$a_i = a_i$ $\qquad$ $b_j = a_j$

GOAL:

Find a simple formula for SUM of all ABOVE OR ON MAIN DIAGONAL DOUBLE SUM (indexed)

$$\boxed{S_{\nabla} = \sum_{1 \le i \le j \le n} a_i a_j}$$

OBSERVATION:

$$\boxed{a_i a_j = a_j a_i}$$

for any i, j

HENCE (PROVE)

$$\boxed{S_{\nabla} = S_{\Delta}}$$

$$\boxed{S_{\Delta} = \sum_{1 \le j \le i \le n} a_i a_j}$$

where S_{Δ} is the SUM of all elements BELOW OR ON main diagonal.

$$\boxed{S_\nabla} = \sum_{\substack{1\le i\le n\\ 1\le j\le n\\ i\le j}} a_i a_j = \boxed{\sum_{\substack{P(i,j)\\ i\le j}} a_i a_j}$$

FOR

$$P(i,j) = (1\le i\le n)\wedge(1\le j\le n) = Q(i)\wedge Q(j)$$

$$\boxed{S_\Delta} = \sum_{\substack{1\le i\le n\\ 1\le j\le n\\ j\le n}} a_i a_j = \boxed{\sum_{\substack{P(i,j)\\ j\le i}} a_i a_j}$$

USE $a_i a_j = a_j a_i$

Evaluate

$$S_\nabla = \sum_{\substack{P(i,j)\\ i\le j}} a_i a_j = \sum_{\substack{P(i,j)\\ i\le j}} a_j a_i = \sum_{\substack{P(i,j)\\ j\le i}} a_i a_j = S_\Delta$$

Re-name $j\to i$, $i\to j$

$P(i,j)=P(j,i)$

WE PROVED

F1 $\boxed{S_\Delta = S_\Delta}$

Evaluate (remember: goal is to find S_Δ)

use (F1) $2S_\Delta = S_\Delta + S_\nabla$

$$= \sum_{\substack{P(i,j)\\ i\le j}} a_i a_j + \sum_{\substack{P(i,j)\\ j\le i}} a_i a_j$$

$Q(i,j)$ $\quad R(i,j)$

$$= \boxed{\sum_{Q(i,j)} a_i a_j + \sum_{R(i,j)} a_i a_j}$$

WE WANT TO COMBINE DOMAINS

PROVE

COMBINED DOMAINES

$$\boxed{\sum_{Q(k)} a_k + \sum_{R(k)} a_k = \sum_{Q(k)\cap R(k)} a_k + \sum_{Q(k)\cup R(k)} a_k}$$

OR

$$\boxed{\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cap K'} a_k + \sum_{k\in K\cup K'} a_k}$$

and the same FOR MULTIPLE SUMS

COUNTING FINITE DOMAINS

FACT

$$\boxed{\text{For any FINITE SETS } A, B \qquad |A \cup B| = |A| + |B| - |A \cap B|}$$

Let's denote

$$K = \{k : Q(k)\}$$

$$K' = \{k : R(k)\}$$

$$|K \cup K'| = |K| + |K'| - |K \cap K'|$$

$$\boxed{|K| + |K'| = |K \cap K'| + |K \cup K'|}$$

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$$\boxed{\underbrace{[k \in K]}_{Q(k)} + [\underbrace{k \in K'}_{R(k)}] = [k \in K \cap K'] + [k \in K \cup K']}$$

predicates

YOU can't ADD PREDICATES
BUT YOU ARE "ADDING"
CHARACTERISTIC FUNCTONS

THIS REALLY MEANS

$$\sum_{Q(k)} a_k + \sum_{R(k)} a_k =$$

$$\sum_{Q(k) \cap R(k)} a_k + \sum_{Q(k) \cup R(k)} a_k$$

COMBINE LIMITS

or

$$\sum_{K} a_k + \sum_{K'} a_k =$$

$$= \sum_{K \cap K'} a_k + \sum_{K \cup K'} a_k$$

$K = K_1 \times \cdot \times K_n$

$K' = K_1' \times \cdot \times K_m'$

COMBINED LIMITS

MULTIPLE SUMS

BACK TO OUR PROBLEM

$$\boxed{2S_{\Delta}} = S_{\Delta} + S_{\nabla}$$

$$= \sum_{\substack{P(ij) \\ i\le j}} a_{ij} + \sum_{\substack{P(i,j) \\ j\le i}} a_{ij}$$

$Q = Q(i,j)$ $\quad$ $R(ij) = R$

$$= \boxed{\sum_{Q\cap R} a_{ij} + \sum_{Q\cup R} a_{ij}}$$

$\boxed{Q} = Q(ij) = P(i,j) \cap (i \le j)$

$\boxed{R} = R(ij) = P(ij) \cap (j \le i)$

We have to evaluate

$\boxed{Q \cap R}$, $\quad$ $\boxed{Q \cup R}$.

EVALUATE $Q \cap R$

$$\boxed{Q \cap R} = (P(i,j) \cap (i \le j)) \cap (P(i,j) \cap j \le i)$$

$$= P(i,j) \cap P(i,j) \cap (i \le j) \cap (j \le i)$$

$$= \boxed{P(i,j) \cap (i = j)}$$

EVALUATE $Q \cup R$ $\quad (A \cap B) \cup (A \cap C) \equiv A \cap (B \cup C)$

$$\boxed{Q \cup R} = (P(i,j) \cap i \le j) \cup (P(i,j) \cap j \le i)$$

$$= P(i,j) \cap (i \le j \cup j \le i)$$ ALWAYS TRUE

$$= \boxed{P(i,j)}$$

$$2 S_{\nabla} = \sum_{Q \cap R} a_i a_j + \sum_{Q \cup R} a_i a_j$$

Remember

Becomes:

$$\boxed{\begin{aligned} P(i,j) = \ & 1 \le i \le n \\ & \cap 1 \le j \le n \end{aligned}}$$

$$\boxed{2S_{\triangledown}} = \sum_{Q \cup R} a_i a_j + \sum_{Q \cap R} a_i a_j$$

$$= \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} a_i a_j + \sum_{\substack{1 \le i \le n \\ 1 \le j \le n \\ i = j}} a_i a_j$$

Prop. proved

$$= \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} a_j\right) + \left(\sum_{i=1}^{n} a_i^2\right)$$

re-name $j \to i$

$$= \left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} a_i^2\right)$$

$$\boxed{S_{\triangledown} = \frac{1}{2}\left(\left(\sum_{i=1}^{n} a_i\right)^2 + \sum_{i=1}^{n} a_i^2\right)}$$

End

Short SOLUTION for

$$\boxed{S_{\nabla} = \sum_{1 \le i \le j \le n} a_i a_j}$$

① Evaluate

by General DISTR.

$$S_{\square} = \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} a_i a_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} a_j\right)$$

② prove $S_{\nabla} = S_{\Delta}$

③ Observe

$$S_{\square} = S_{\nabla} + S_{\Delta} - \sum_{i=1}^{n} (a_i)^2$$

$$= 2 S_{\nabla} - \sum_{i=1}^{n} (a_i)^2$$

④ SOLVE :

$$S_{\nabla} = \frac{1}{2}\left(S_{\square} + \sum_{i=1}^{n} (a_i)^2 \right)$$

$$S_{\nabla} = \frac{1}{2}\left(\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} a_j\right) + \sum_{i=1}^{n} (a_i)^2 \right)$$

$$\boxed{S_{\nabla} = \frac{1}{2}\left(\left(\sum_{i=1}^{n} a_i\right)^2 + \sum_{i=1}^{n} (a_i)^2 \right)}$$

$\boxed{\text{NEW PROBLEM}}$

Given $\{a_n\}, \{b_n\}_{n \in \mathbb{N}}$

EVALUATE

$$S = \sum_{1 \le j < k \le n} (a_k - a_j)(b_k - b_j)$$

$$\boxed{S = \sum_{\substack{1 \le j \le n \\ 1 \le k \le n \\ j < k}} (a_k - a_j)(b_k - b_j)}$$

$$P(j,k) = (1 \le j \le n) \wedge (1 \le k \le n)$$

$$= P_1(j) \wedge P_2(k)$$

(1)

$$\boxed{S = \sum_{\substack{P(j,k) \\ j < k}} (a_k - a_j)(b_k - b_j)}$$

Observe $P(j,k) = P(k,j)$

OBSERVE

① EXCHANGE j and k in S (re-name)

$$\boxed{S = \sum_{\substack{P(j,k) \\ k<j}} (a_j - a_k)(b_j - b_k)}$$

$k<j$ because $P(j,k) = P(k,j)$

② EVALUATE

$$\boxed{(a_j - a_k)(b_j - b_k)} = -(a_k - a_j)(-(b_k - b_j)) = \boxed{(a_k - a_j)(b_k - b_j)}$$

③ S BECOMES

②

$$\boxed{S = \sum_{\substack{P(j,k) \\ k<j}} (a_k - a_j)(b_k - b_j)}$$

ADD ① + ②

$$2S = \sum_{\substack{P(jk)\\ j \le k}} (a_k - a_j)(b_k - b_j) + \sum_{\substack{P(jk)\\ k \le j}} (a_k - a_j)(b_k - b_j)$$

$j=k$ gives 0 term

USE

$$\boxed{\sum_{K} a_k + \sum_{K'} a_k = \sum_{K \cap K'} a_k + \sum_{K \cup K'} a_k}$$

Evaluate

$\boxed{K \cap K'}$ = ~~the~~

$= P(jk) \cap (j \le k) \cap P(jk) \cap (k \le j)$

$= \boxed{P(jk) \cap k = j}$

Observe

$k = j$ gives term 0

so we can sum over $j \le k$, $k \le j$

$$\sum_{\substack{P(jk)\\ j<k}} 0 = \sum_{\substack{P(jk)\\ j \le k}} 0$$

$(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$ 112

EVALUATE

$$\boxed{K \cup K'} = (P(j,k) \cap j \le k) \cup (P(j,k) \cap k \le j)$$

ALWAYS TRUE

$$= P(j,k) \cap (j \le k \cup k \le j)$$

USED

$$= \boxed{P(j,k)}$$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributivity
+ Fact that
$j \le k \cup k \le j$
TRUE for
all j, k

We re-write our 2S as follows

$$\boxed{2S} = \sum_{\substack{P(j,k) \\ j \le k}} (a_k - a_j)(b_k - b_j) + \sum_{\substack{P(j,k) \\ k \le j}} (a_k - a_j)(b_k - b_j)$$

K K'

$$= \sum_{\substack{P(j,k) \\ j = k}} (a_k - a_j)(b_k - b_j) + \sum_{P(j,k)} (a_k - a_j)(b_k - b_j)$$

$K \cap K'$ $K \cup K'$

Observe

$$\sum_{\substack{P(jk)\\ j=k}} (a_k - a_j)(b_k - b_j) = 0$$

$$\boxed{2S = \sum_{P(j,k)} (a_k - a_j)(b_k - b_j)}$$

$P(j,k) = P(k,j)$

Expend now

$$\boxed{(a_k - a_j)(b_k - b_j)} =$$

$a_j b_k = b_k a_j$

$$= a_k b_k - a_j b_k - a_k b_j + a_j b_j$$

DO SUMMATION

$$= \cancel{a_k b_k + a_j b_j - 2 a_k b_j}$$

$$= \cancel{2 a_k b_k - 2 a_k b_j}$$

re-name $j \leftrightarrow k$

$$\sum(\ )(\ ) = \sum_P a_k b_k + \sum_P a_j b_j - 2 \sum a_k b_j$$

$$\boxed{2S = 2\sum_{P(jk)} a_k b_k - 2\sum_{P(jk)} a_k b_j}$$

Use distributivity for

$$\boxed{\sum_{P(j,k)} a_k b_j} = \sum_{\substack{1\leq j \leq n \\ 1\leq k \leq n}} a_k b_j$$

$$= \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right)$$

re-name $j \to k$ in second sum

$$= \boxed{\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)}$$

we evaluate

$$\boxed{\sum_{P(jk)} a_k b_k}$$

separetly

NOT clear how to use distributivity!

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$$\boxed{\sum_{P(j,k)} a_k b_k} = \sum_{\substack{1 \le j \le n \\ 1 \le k \le n}} a_k b_k$$

DEFINITION $$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \underbrace{a_k b_k}_{\text{constant with respect to } j} \right)$$

$$= \sum_{k=1}^{n} \left(a_k b_k \underbrace{\sum_{j=1}^{n} 1}_{= n} \right)$$

$$= \sum_{k=1}^{n} a_k b_k \cdot n \longrightarrow \text{constat with } k$$

$$= \boxed{n \sum_{k=1}^{n} a_k b_k}$$

$$2S = 2\sum_{P(i,j)} a_k b_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

$$\boxed{S = n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)}$$

① DOUBLE SUM → SIMPLE SUMS

$$\boxed{\sum_{\substack{1\le k\le n\\ 1\le j\le n\\ j<k}} (a_k - a_j)(b_k - b_j) = n\sum_{k=1}^{n} a_k b_k - \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)}$$

USE IT TO EVALUATE RELATIONSHIP BETWEEN

$$\sum_{k=1}^{n} a_k b_k \quad \text{and} \quad \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

CHEBYSHEV'S INEQUALITIES

RE-WRITE (1)

$$\boxed{\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) = n\sum_{k=1}^{n} a_k b_k - \sum_{\substack{P(k,j) \\ j<k}} (a_k - a_j)(b_k - b_j)}$$

ASSUME

(C1)

$a_1 \le a_2 \le a_3 \le \quad \le a_n$

$b_1 \le b_2 \le b_3 \le \quad \le b_n$

UNDER (C1) $(a_k - a_j)$ for $j<k$ is POSITIVE

$(a_2 - a_1)$

$(a_3 - a_1)$

as is $(b_k - b_j)$ so is $\sum (a_k - a_j)(b_k - b_j)$ POSITIVE

and we get

(1) $$\boxed{\left(\sum a_k\right)\left(\sum b_k\right) \le n \sum_{k=1}^{n} a_k b_k}$$

WHEN (C1) HOLDS

for

$a_1 \le a_2 \le \quad \le a_n$

$b_1 \le b_2 \quad \le b_n$

ASSUME

C2 $a_1 \le a_2 \quad \le a_m$

$b_1 \geq b_2 \quad \geq b_m$

C2 is a CONDITION FOR

$\sum (a_k - a_j)(b_k - b_j)$ TO BE NEGATIVE

$\quad + \qquad\qquad -$

as it has all negative terms

and $-\sum(a_k - a_j)(b_k - b_j)$ is POSITIVE

and we GET

$$\text{(2)} \quad \boxed{\left(\sum a_m\right)\left(\sum b_m\right) \geq n \sum_{k=1}^{n} a_k b_k}$$

for $a_1 \le a_2 \quad \le a_m$

$b_1 \geq b_2 \quad \geq b_m$