$\boxed{\text{INFINITE CALCULUS}}$

$\boxed{\text{INTEGRATION}}$

REMINDER

$$\boxed{\begin{array}{l} D : R^R \longrightarrow R^R \\ D f(x) = g(x) = f'(x) \end{array}}$$

$\boxed{\text{D is PARTIAL}}$

Dom D = all differentiable functions

D is not 1-1 ; D(c) = 0 all $c \in R$.

so INVERSE function does not exist

BUT we define a $\boxed{\text{REVERSE}}$ process to DIFFENNTIATION that is called $\boxed{\text{INTEGRATION}}$

① we define a notion of a $\boxed{\text{PRIMITIVE FUNCTION}}$

② use it to give a general definition of $\boxed{\text{INDEFINITE INTEGRAL}}$

DEFINITION

A function $\boxed{F(x) = F}$ such that $\boxed{DF = DF(x) = F'(x) = f(x)}$

is called a **PRIMITIVE FUNCTION** of $f(x)$, or simply a

PRIMITIVE of $f(x) = f$

Shortly

> F is a PRIMITIVE OF f
> iff $\quad DF = f$
> $(F' = f)$

F is a PRIMITIVE of f iff f is obtained from F by differentiation

> The process of finding primitive of f is called integration

$\boxed{\text{PROBLEM}}$: given f find ALL primitive functions of f (if exist)

FUNDAMENTAL THEOREM (of diff and INTEGRAL calculus)

The difference of two primitives $F_1(x)$, $F_2(x)$ of the same function $f(x)$ is $\boxed{\text{CONSTANT}}$

i.e

$$\boxed{F_1(x) - F_2(x) = C}$$

for any F_1, F_2 such that

$$D F_1(x) = f(x) \text{ and } D F_2(x) = f(x)$$

IT MEANS

1. From any primitive function $F(x)$ we obtain all the others in the form $\boxed{F(x) + C}$ (suitable C)
2. For every value of C, the ~~expression~~ function $\boxed{F_1(x) = F(x) + C}$ represents a $\boxed{\text{PRIMITIVE of } f}$

DEFINITION of INDEFINITE INTEGRAL AS a GENERAL FORM of A PRIMITIVE FUNCTION of f

$$\boxed{\int f(x)\,dx = F(x) + C} \qquad \boxed{C \in \mathbb{R}}$$

where $F(x)$ is any primitive of f

i.e $\boxed{DF(x) = f(x)}$ $\boxed{F' = f}$
short

PROOF of the FUNDAMENTAL THEOREM of differential and Integral calculus.

② we prove that if $F(x)$ is primitive to $f(x)$, so is $F(x)+C$; i.e

$D(F(x)+C) = f(x)$, when $DF(x) = f(x)$

① $F_1(x) - F_2(x) = C$ i.e

from any primitive $F(x)$ we obtain all others in the form $F(x)+C$

(2) $G(x) = F(x) + C$

$$\boxed{D(F(x)+c)} = \lim_{h\to 0} \frac{(F(x+h)+c) - (F(x)+c)}{h}$$

$$= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \boxed{f(x)}$$

as $F(x)$ is a PRIMITIVE of $f(x)$.

$F_1' = f$, $F_2' = f$

(1) CONSIDER: $F_1(x) - F_2(x) = G(x)$

want to show that $G(x) = C$ all $x \in \mathbb{R}$

Evaluate $G'(x) = D\,G(x)$

$$D(G(x)) = \lim_{h\to 0} \frac{(F_1(x+h) - F_2(x+h)) - (F_1(x) - F_2(x))}{h}$$

$$= \lim_{h\to 0} \left(\frac{F_1(x+h) - F_1(x)}{h} - \frac{F_2(x+h) - F_2(x)}{h} \right)$$

Both limits exist, as F_1, F_2 primitive of f.

$$= \lim_{h\to 0} \frac{F_1(x+h) - F_1(x)}{h} - \lim_{h\to 0} \frac{F_2(\ ) - F_2(x)}{h}$$

$$= f(x) - f(x) = 0 \quad \text{all } x \in \mathbb{R}.$$

$F_1(x) - F_2(x) = G(x)$

and $\boxed{G'(x) = 0}$ for all $x \in R$ Intuition

But the function whose derivative is everywhere zero must have a graph whose tangent is everywhere parallel to x-axis; i.e must be constant; and therefore we have

$\boxed{G(x) = C}$

Formal proof. Apply the $\boxed{\text{MEAN VALUE THEOREM}}$ to $G(x)$ i.e

$G(x_2) - G(x_1) = (x_2 - x_1) G'(\xi) \quad x_1 < \xi < x_2$

but $G'(x) = 0$ for all x, hence $G'(\xi) = 0$

and

$G(x_2) - G(x_1) = 0$ for <u>any</u> x_1, x_2

i.e $G(x_2) = G(x_1)$ all x_1, x_2 i.e

$\boxed{G(x) = C.}$

This (① + ②) justifies the definition

INDIFINITE INTEGRAL.

$$\boxed{\int f(x) = F(x) + C \quad , \quad DF(x) = F'(x) = f(x)}$$

FINITE CALCULUS

$\Delta : R^R \longrightarrow R^R$, for any $f \in R^R$

$\Delta f(x) = f(x+1) - f(x)$

Δ TOTAL on R^R

Δ is DIFFERENCE OPERATOR

defined for <u>all</u> $f : R \to R$

Δ is a ~~partial~~ ~~TOTAL~~ total function on R^R.

INVERSE TO Δ d.n.e !

Remark :

Δ is not 1−1 function

NOT ONLY ONE EXAMPLE

~~Take~~ $f_1(x) = c_1$ $\quad f_2(x) = c_2$, $c_1 \neq c_2$, all x

i.e $f_1 \neq f_2$ we have

$$\Delta f_1(x) = f_1(x+1) - f_1(x) = c_1 - c_1 = 0$$

$$\Delta f_2(x) = f_2(x+1) - f_2(x) = c_2 - c_2 = 0$$

$\Delta f_1 = \Delta f_2$ (for $f_1 \neq f_2$)

Q1. Do we have an **REVERSE** operation to Δ as we did for D ?

Ans. **YES !** We proceed as in Infinite calculus

DEFINITION

A function $F=F(x)$ is **FINITELY PRIMITIVE** of $f=f(x)$

iff

$$\Delta F(x) = f(x) \qquad \text{all } x \in R$$

$$\Delta F = f.$$

The process of finding finitely primite (FP) function of $f=f(x)$ is called a FINITE INTEGRATION

PROBLEM: Given f, find all FINITELY PRIMITEVE (FP) functions of $f(x)$.

FUNDAMENTAL THEOREM (of FINITE CALCULUS)

The difference of two FPrimitives $F_1(x)$, $F_2(x)$ of the same function $f(x)$ is a function $C(x)$, such that $C(x+1) = C(x)$ i.e

$$F_1(x) - F_2(x) = C(x)$$

and $C(x+1) = C(x)$ for all $x \in R$

It means

① From any FP function $F(x)$ of $f(x)$ we obtain all others in the form $F(x) + C(x)$, for

$C: R \to R$ $C(x+1) = C(x)$

② For every function $C(x)$, such that $C(x+1) = C(x)$, the function $F_1(x) = F(x) + C(x)$ is a FPrimitive of $f(x)$

PROOF of FUNDAMENTAL THEOREM

For
$\Delta F_1 = f$
$\Delta F_2 = f$

① Consider

$$\boxed{F_1(x) - F_2(x) = C(x)}$$

we want to show that

$$\boxed{C(x+1) = C(x)}$$

Evaluate

$$\Delta C(x) = \boxed{C(x+1) - C(x)} = \Delta\big(F_1(x) - F_2(x)\big)$$

$$= \big(F_1(x+1) - F_2(x+1)\big) - \big(F_1(x) - F_2(x)\big)$$

$$= \big(F_1(x+1) - F_1(x)\big) - \big(F_2(x+1) - F_2(x)\big)$$

$$= f(x) - f(x) = \boxed{0}$$

i.e $C(x+1) - C(x) = 0$

$$\boxed{C(x+1) = C(x)}$$

$$\begin{cases} \Delta F(x) = f(x) \\ C(x+1) = C(x) \end{cases}$$

② Let $F_1(x) = F(x) + C(x)$ and
we prove that $F_1(x)$ is F PRIMITIVE of f

$$\boxed{\Delta F_1(x)} = \big(F(x+1) + C(x+1)\big) - \big(F(x) + C(x)\big)$$

$$= F(x+1) - F(x) + 0 = \Delta F(x) = \boxed{f(x)}$$

yes.

DEFINITION OF INDEFINITE SUM

as a GENERAL FORM of a FINITELY PRIMITIVE function of $f=f(x)$

$$\boxed{\begin{gathered}\sum g(x)\,\delta(x) = f(x) + C(x) \\ \text{iff} \\ g(x) = \Delta f(x) \quad \text{and} \quad C(x+1) = C(x)\end{gathered}}$$

for $g: R \to R$; $f(x): R \to R$, $C: R \to R$

Remark : in particular case

$$\boxed{C(x) = C \quad \text{for all} \quad x \in R}$$

(as in the case of Indefinite Integral)

$$C(x+1) = C = C(x)$$

EXAMPLE OF A "CONSTANT" function $C = p(x)$ under Δ

$p(x) = \sin 2\pi x$ (PERIODIC function)

$p(x+1) = \sin(2\pi(x+1))$

$= \sin(2\pi x + 2\pi) = p(x)$ all $x \in \mathbb{R}$

INFINITE CALCULUS:

DEFINITE INTEGRAL

$$\boxed{\int_a^b g(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)}$$

where $\boxed{f'(x) = g(x)}$

FINITE CALCULUS:

DEFINITE SUM

FINITE INTEGRATION

$$\boxed{\sum\nolimits_a^b g(x)\,\delta x = f(x)\Big|_a^b = f(b) - f(a)}$$

DEFINITION $\boxed{\text{where } \Delta f(x) = g(x)}$

Definite SUM definition

$$\boxed{\sum_a^b g(x)\,\delta x = f(x)\Big|_a^b = f(b) - f(a)}$$

for $f(x)$ such that

$$\boxed{g(x) = \Delta f(x)}$$

$\Delta f = g$

What is the MEANING of

$\sum_a^b g(x)\,\delta x$? "INTEGRAL"

$\sum_{x=1}^{5} f(x)$

$g(x) = \Delta f(x) = f(x+1) - f(x)$

TAKE $b = a$

$$\boxed{\sum_a^a g(x)\,\delta x} = f(a) - f(a) = 0$$

TAKE $b = a+1$

$$\boxed{\sum_a^{a+1} g(x)\,\delta x} = f(a+1) - f(a) = \Delta f(a) = g(a)$$

$$\sum_a^a g(x)\delta_x = 0$$

$$\boxed{\sum_a^{a+1} g(x)\delta_x = g(a)}$$

Evaluate

$$\sum_a^{a+2} g(x)\delta_x \overset{def}{=} f(a+2) - f(a)$$

Consider

$$\boxed{\sum_a^{a+2} g(x)\delta_x - \sum_a^{a+1} g(x)\delta_x} \overset{def}{=}$$

$$= f(a+2) - f(a) - (f(a+1) - f(a))$$

$$= f(a+2) - f(a) - f(a+1) + f(a)$$

$$= f(a+2) - f(a+1) = g(a+1)$$

$$\boxed{\sum_a^{a+2} g(x)\delta_x} = \sum_a^{a+1} g(x)\delta_x + g(a+1)$$

$$= \boxed{g(a) + g(a+1)}$$

We proved

$$\boxed{\begin{aligned} &\sum_a^{a+1} g(x)\,\delta_x = g(a) \\ &\sum_a^{a+2} g(x)\,\delta_x = g(a) + g(a+1) \end{aligned}}$$

Evaluate

$$\sum_a^{a+3} g(x)\,\delta_x \overset{\text{DEF}}{=} f(a+3) - f(a)$$

COMPUTE

$$\boxed{\sum_a^{a+3} g(x)\,\delta_x - \sum_a^{a+2} g(x)\,\delta_x} \overset{\text{def}}{=}$$

$$= f(a+3) - f(a) - f(a+2) + f(a)$$

$$= f(a+3) - f(a+2) = g(a+2)$$

$$\boxed{\sum_a^{a+3} g(x)\,\delta_x} = \sum_a^{a+2} g(x)\,\delta_x + g(a+2)$$

$$= \boxed{g(a) + g(a+1) + g(a+2)}$$

GUESS (proof by math. induction over k)

$b \geq a$

$$\boxed{\sum_a^{a+k} g(x)\delta_x = g(a)+g(a+1)+\cdots g(a+k-1)}$$

$a+k=b$

$a+k-1=b-1$

$f(b)-f(a)$ $\|$

INDEFINITE SUM

$$\sum_a^b g(x)\delta_x = \sum_{a \leq k < b} g(k)$$

NORMAL SUM

$f(b)-f(a)$ where $\Delta f(x) = g(x)$

$$= \sum_{k=a}^{b-1} g(k)$$

NORMAL SUM

Relationship between INDEFINITE $\Leftrightarrow$ NORMAL

$b < a$

DEFINITE SUM PROPERTIES

① $$\boxed{\sum_a^b g(x)\,\delta x} = f(b) - f(a)$$

$$= -\left(f(a) - f(b)\right)$$

$$= \boxed{-\sum_b^a g(x)\,\delta x}$$

② $$\boxed{\sum_a^b g(x)\,\delta x + \sum_b^c g(x)\,\delta x = \sum_a^c g(x)\,\delta x}$$

For all $a, b, c \in \mathbb{Z}$.

Reminder

$$\Delta\left(x^{\underline{m}}\right) = m\,x^{\underline{m-1}}$$

$$\boxed{\sum_{0 \le k < n} k^{\underline{m}} = \left.\frac{k^{\underline{m+1}}}{m+1}\right|_0^n = \frac{n^{\underline{m+1}}}{m+1}}$$

$n, m \geq 0$

$$\int_0^n x^m\,dx = \left.\frac{x^{m+1}}{m+1}\right|_0^n = \frac{n^{m+1}}{m+1}$$

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PROBLEM:

FIND $$\boxed{\sum_{k=0}^{n-1} k^{\underline{m}}}$$

USE:

$$\Delta x^{\underline{m}} = m x^{\underline{m-1}}$$

SOLUTION:

$$\sum_{k=0}^{n-1} k^{\underline{m}} \overset{2}{=} \sum\nolimits_0^n x^{\underline{m}}\,\delta x = \sum_{0\le k<n} g(k)$$

$$= \frac{x^{\underline{m+1}}}{m+1}\Bigg|_0^n = \frac{n^{\underline{m+1}}}{m+1}$$

Used:

1 $$\Delta\left(\frac{x^{\underline{m+1}}}{m+1}\right) = \frac{1}{m+1}\Delta x^{\underline{m+1}} = \frac{m+1}{m+1}\cdot x^{\underline{m}}$$

$$= x^{\underline{m}}$$

2 THM

$$\sum\nolimits_a^b g(x)\,\delta x = \sum_{a\le k<b} g(k) = \sum_{k=a}^{b-1} g(k)$$

$$\overset{''}{f(b)-f(a)}$$

$$\Delta f(x) = g(x)$$

$$\cancel{\Delta g(k)} = \sum k^{\underline{m}}$$

In particular, when $m=1$

$$\boxed{k^{\underline{1}} = k} \qquad \boxed{x^{\underline{m}} = x(x-1)\dots(x-m+1)}$$

Problem: evaluate $\sum_{k=0}^{n-1} k = \sum_{0 \le k < n} k$
use "integration".

$$\sum_{k=0}^{n-1} k = \sum_{k=0}^{n-1} k^{\underline{1}} = \sum_{0 \le k < n} k^{\underline{1}}$$

THM

$$= \sum_0^n x^{\underline{1}} \delta x = \frac{x^{\underline{2}}}{2} \Big|_0^n$$

$$= \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

$$n^{\underline{2}} = n(n-1)\dots(n-2+1) = n(n-1)$$

$$\boxed{\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}}$$

FACT 1

$$\boxed{K^2 = K^{\underline{2}} + K^{\underline{1}}} \qquad x^{\underline{m}} = x(x-1)\ldots(x-m+1)$$

Proof :

$$K^{\underline{2}} = K(K-2+1) = K(K-1)$$

$$K^{\underline{1}} = K$$

$$K^{\underline{2}} + K^{\underline{1}} = K(K-1) + K = K(K-1+1) = K^2$$

PROBLEM : Evaluate $\boxed{\sum_{K=0}^{n-1} K^2}$

SOLUTION : USE THM + FACT 1

$$\boxed{\sum_{K=0}^{n-1} K^2} \overset{F1}{=} \sum_{0 \le k < n} \left(k^{\underline{2}} + k^{\underline{1}}\right) = \sum_{0 \le k < n} k^{\underline{2}} + \sum_{0 \le k < n} k^{\underline{1}} =$$

$$\overset{THM}{=} \sum_0^n x^{\underline{2}} \delta x + \sum_0^n x^{\underline{1}} \delta x$$

$$= \frac{x^{\underline{3}}}{3} \Bigg|_0^n + \frac{x^{\underline{2}}}{2} \Bigg|_0^n = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}$$

$$= \frac{1}{3}\left(n(n-1)(n-2)\right) + \frac{1}{2}\left(n(n-1)\right) = \boxed{\frac{1}{3} n\left(n-\frac{1}{2}\right)(n-1)}$$

$$1/3\, n(n-1)(n-2+3/2)$$

FACT

$$\boxed{k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}}$$

evaluate!

$$\boxed{\sum_{a \le k < b} k^3 = \frac{k^{\underline{4}}}{4} + 3\cdot\frac{k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2}\Bigg|_a^b}$$

Homework P1

PROVE

$$\boxed{(x+y)^{\underline{2}} = x^{\underline{2}} + 2x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}}}$$

$$x^{\underline{3}} = x(x-1)(x-2)$$

$$x^{\underline{2}} = x(x-1)$$

$$x^{\underline{1}} = x$$

$$x^{\underline{0}} = 1$$

$$x^{\underline{2}} = \frac{x^{\underline{3}}}{x-2}$$

$$x^{\underline{1}} = \frac{x^{\underline{2}}}{x-1}$$

$$x^{\underline{0}} = \frac{x^{\underline{1}}}{x}$$

follows a pattern

$$x^{\underline{-1}} = \frac{1}{x+1}$$

definition