$\boxed{\text{DEFINITION}}$ (of negative exponent falling powers)

- $x^{\underline{-1}} = \frac{1}{x+1}$
- $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$
- $x^{\underline{-3}} = \frac{1}{(x+1)(x+2)(x+3)}$

GENERAL

$$\boxed{x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\cdots(x+m)}} \quad m > 0$$

HOMEWORK :

$$x^{m+n} = x^n \cdot x^m$$

PROVE

$$\boxed{x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}}$$

HOMEWORK

PROVE

we proved for $m \geq 0$

$$\boxed{\Delta x^{\underline{m}} = m x^{\underline{m-1}}}$$

for $m<0$

Example

$$\boxed{\Delta x^{\underline{-2}}} = \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)}$$

$$= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)}$$

$$= \boxed{-2\, x^{\underline{-3}}}$$

FACT

$$\boxed{\sum_a^b x^{\underline{m}}\, \delta x = \left.\frac{x^{\underline{m+1}}}{m+1}\right|_a^b}$$

all $m \neq -1$

What about case $m=-1$?

CASE $m=-1$

INFINITE INTEGRAL

$$\int_a^b x^{-1}\,dx = \int_a^b \frac{1}{x}\,dx = \ln x \Big|_a^b$$

We want to have a FINITE Analog

$$\boxed{x^{\underline{-1}} = \frac{1}{x+1} \quad \Big| \quad \Delta f = f(x+1) - f(x)}$$

TAKE

$$\boxed{f(x)} = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{x} = \boxed{\sum_{k=1}^{x} \frac{1}{k} = H_x}$$

$$\Delta f(x) = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{x} + \frac{1}{x+1}\right) - \left(\frac{1}{1} + \cdots \frac{1}{x}\right)$$

$$= \frac{1}{x+1}$$

CASE $m=-1$

$$\boxed{\sum_a^b x^{\underline{-1}}\,\delta_x = \sum_a^b \frac{1}{x+1}\,\delta_x = H_x \Big|_a^b}$$

we will prove (Ch 9) that for LARGE x

$$\boxed{H_x - \ln_x \approx 0.577 + \tfrac{1}{2x}} \to 0$$

$H_x \sim \ln_x$ as do $\int_a^b$ and $\sum_a^b$.

THM
SUMS OF FALLING POWERS:

$$\sum_a^b x^{\underline{m}} \delta x = \begin{cases} \left.\dfrac{x^{\underline{m+1}}}{m+1}\right|_a^b & m \neq 1 \\ \left. H_x \right|_a^b & m = -1 \end{cases}$$

$\underline{\text{thm}}$ //

$\sum_a^{b-1} x^{\underline{m}}$

all $m \in \mathbb{Z}$.

and $\int_a^b \frac{1}{x} dx = \ln x \Big|_a^b$ is SIMILAR (for large x) $\sum_a^b x^{\underline{-1}} = H_x \Big|_a^b$

MORE SIMILARITIES

We know

$$(e^x)' = e^x$$

$$\boxed{De^x = e^x}$$

$Df = f$
FOR $f = e^x$

Q. Which function $f = f(x)$ has a similar property for Δ? i.e

$$\boxed{\Delta f(x) = f(x)}$$

$$\Delta f(x) = f(x+1) - f(x) = f(x)$$

f is such that: $\boxed{f(x+1) = 2f(x)}$

Reccurence!

EXAMPLE of SOLUTION

$$\boxed{f(x) = 2^x}$$

$$f(x+1) = 2^{x+1} = 2 \cdot 2^x = 2f(x)$$

$f: \mathbb{Q} \to \mathbb{R}$

we proved

$$\boxed{\Delta(2^x) = 2^x}$$

$$(e^x)' = e^x$$

Which is a formula for Δf,

where $f(x) = c^x$ $\quad f: R \to R$

$$\Delta(c^x) = c^{x+1} - c^x = c \cdot c^x - c^x$$
$$= c^x(c-1)$$

Difference:

$$\boxed{\Delta(c^x) = (c-1)c^x}$$

$c \in N^+$

"derivative"

$$\boxed{\sum_a^b c^x \delta_x = \frac{c^x}{(c-1)} \Big|_a^b \qquad c \neq 1}$$

"antiderivative"

anti-difference

We proved a Thm

$$\boxed{\sum_{a\le k<b} c^k} = \sum_a^b c^x \delta x = \frac{c^x}{(c-1)}\Bigg|_a^b = \boxed{\frac{c^b - c^a}{c-1}} \quad c \ne 1$$

Geometric progression

GENERAL FORMULA FOR GEOMETRIC PROGRESSION

$$\boxed{\sum_{a\le k<b} c^k = \frac{c^b - c^a}{(c-1)}} \quad c \ne 1$$

$$\sum_{k=a}^{b-1} c^k = \frac{c^b - c^a}{(c-1)}$$

$\boxed{\text{INFINITE}}$: "chain rule"

$$D f(g(x)) = Df \cdot Dg(x)$$

$\boxed{\text{FINITE}}$: NO such rule

Can't relate $\Delta f(g(x))$ to $\Delta g(x)$

INFINITE

$$\boxed{D(u \cdot v) = u Dv + v Du}$$

Integration by parts

$$\boxed{\int u Dv = u \cdot v - \int v Du}$$

Does it have an ANALOG for Δ ?

i.e $\Delta(uv) = u \Delta v + v \Delta u$

and $\sum u \Delta v = uv - \sum v \Delta u$?

NO exactly, but CLOSE! change here.

$$\Delta(uv) \overset{?}{=} u\Delta v + v\Delta u$$

EVALUATE

$$\Delta(u(x)\cdot v(x)) = u(x+1)v(x+1) - u(x)v(x)$$

$$= u(x+1)v(x+1) \overbrace{- u(x)v(x+1) + u(x)v(x+1)}^{0} - u(x)v(x)$$

GROUP

$$= u(x)v(x+1) - u(x)v(x)$$

$$+ u(x+1)v(x+1) - u(x)v(x+1)$$

$$= u(x)(v(x+1) - v(x)) + v(x+1)(u(x+1) - u(x))$$

$$= u(x)\Delta v(x) + v(x+1)\Delta u(x)$$

$$'' \boxed{Ev = v(x+1)}$$

SHIFT OPERATOR

We PROVED

$$\boxed{\Delta(u\cdot v) = u\cdot\Delta v + Ev\,\Delta u}$$

$$\sum \Delta u\cdot v = \sum u\Delta v + \sum Ev\Delta u$$

SUMMATION BY PARTS

$Ev = v(x+1)$

$$\boxed{\sum u\,\Delta v = u\cdot v - \sum Ev\,\Delta u}$$

$$\boxed{\sum_a^b u\,\Delta v = u\cdot v\Big|_a^b - \sum_a^b Ev\,\Delta u}$$

INTEGRATION

$f = x \quad g' = e^x$

$$\boxed{\int x e^x\,dx} = xe^x - \int 1\cdot e^x\,dx = \boxed{e^x(x-1)+c}$$

SUMMATION

$$\boxed{\sum x 2^x\,\delta x} = x2^x - \sum 2^{x+1}\,\delta x = \boxed{x2^x - 2^{x+1} + C}$$

$C(x) = C(x+1)$

parts again !

or

$$\boxed{\sum cf = c\sum f}$$

$u(x) = x, \quad v(x) = 2^x, \quad Ev = 2^{x+1}$

$\Delta u = 1, \quad \Delta v(x) = 2^x$

FACT $\quad \Delta 2^{x+1} = 2^{x+1}$

IN PARTICULAR EVALUATE

$$\boxed{\sum_{k=0}^{n} k2^k} = 1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + \dots n\cdot 2^n$$

$\Delta 2^x = 2^x$

$\sum 2^x \delta_x = 2^x + c$

$u = x$

$\Delta v = 2^x$

$v = 2^x$

$$\sum_{k=0}^{n} k2^k \overset{\text{(11)}}{=} \sum_{0}^{n+1} x\, 2^x \delta_x$$

$$= \left(x2^x - 2^{x+1}\right)\Big|_0^{n+1}$$

$$= \left((n+1)2^{n+1} - 2^{n+2}\right) - \left(0\cdot 2^0 - 2\right)$$

$$= (n+1)2^{n+1} - 2\cdot 2^{n+1} + 2$$

$$= (n+1-2)2^{n+1} + 2 = (n-1)2^{n+1} + 2$$

! $$\boxed{\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2}$$!

USE FINITE CALCULUS TO EVALUATE: $\boxed{\sum_{k=0}^{n-1} k H_k}$ "SUM" by parts

Analog:

$$\int \underset{Dv}{x} \underset{u}{\ln x}\, dx = \underset{v \cdot u}{\frac{x^2}{2} \cdot \ln x} - \underset{v \cdot Du}{\int \frac{x^2}{2} \cdot \frac{1}{x}\, dx}$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2}\, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2}$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

USE $\boxed{\sum u \Delta v = uv - \sum E v \Delta u}$

$Ev = v(x+1)$

$$\sum_{k=0}^{n-1} k H_k = \boxed{\sum_{0 \le x < n} \underset{u \cdot \Delta v}{x H_x}\, \delta x}$$

$H_x = u(x)$ $\quad x = \Delta v = x^{\underline{1}}, \quad v(x) = \frac{x^{\underline{2}}}{2}$

USE

$$\boxed{\sum u \Delta v = u \cdot v - \sum E v \Delta u}$$

$$E v = v(x+1)$$

$$\sum_{0 \le x < n} \underbrace{x}_{\Delta v} \underbrace{H_x}_{u} \delta_x = u \cdot v - \sum E v \Delta u$$

$$\Delta v(x) = x = x^{\underline{1}}, \quad \boxed{v(x) = \frac{x^{\underline{2}}}{2}} \qquad \boxed{u(x) = H_x}$$

$$\Delta u(x) = \Delta H_x = x^{\underline{-1}} \qquad \boxed{\Delta u(x) = x^{\underline{-1}}}$$

$$v(x) = \frac{x^{\underline{2}}}{2}, \quad v(x+1) = \frac{(x+1)^{\underline{2}}}{2}$$

$$\boxed{E v = \frac{(x+1)^{\underline{2}}}{2}}$$

$$\boxed{\sum_{k=0}^{n-1} k H_k} = \sum_{0 \le x < n} x H_x \delta_x = \sum_0^n x H_x \delta_x$$

$$= \left(\frac{x^{\underline{2}}}{2} \cdot H_x - \sum \frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}} \delta_x \right) \Bigg|_0^{n}$$

Evaluate : $\frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}}$

Evaluate:

$$\boxed{\frac{(x+1)^{\underline{2}}}{2} \cdot x^{\underline{-1}}} = \frac{1}{2}\, x(x+1) \cdot \frac{1}{(x+1)} = \frac{1}{2}\, x$$

$$= \frac{1}{2}\, x^{\underline{1}}$$

$$\boxed{\sum_{k=0}^{n-1} k H_k} = \sum_{0<x<n} x\, H_x\, \delta x$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2}\sum x^{\underline{1}}\, \delta x\right)\Bigg|_0^n$$

$$= \left(\frac{x^{\underline{2}}}{2} H_x - \frac{1}{2}\,\frac{x^{\underline{2}}}{2}\right)\Bigg|_0^n$$

$$= \frac{x^{\underline{2}}}{2}\left(H_x - \frac{1}{2}\right)\Bigg|_0^n = \boxed{\frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right)}$$

$$\boxed{\sum_{k=0}^{n-1} k H_k = \frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right)}$$