cse547, math547
DISCRETE MATHEMATICS

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LECTURE 13

CHAPTER 5
Binomial Coefficients

Basic Definitions

Definition

For any $n, k \in N, \;\; k \geq 0, \;\; k \leq n$ we define

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)\;\dots\;2\cdot 1}$$

Observe that

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

Combinatorial interpretation

$\binom{n}{k}$ reads: "n choose k"

$\binom{n}{k}$ denotes a number of ways to choose k-element subset from an n-element set

Combinatorial Interpretation

The **number** of ways to **choose** a k-element subset from an n-element set is

$$\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k(k-1)\ \dots\ 2\cdot 1}$$

Proof We carry the proof in two steps

Step 1: we find the **number** of k-element, 1-1 **sequences** formed out of any n-element set

By definition, all **sequences** of length k formed from n-element set are all possible functions

$$f:\ \{1,2,\ \dots\ ,k\} \longrightarrow \{a_1,\ \dots\ ,a_n\}$$

We know, by the Counting Functions Theorem that and there are n^k of them

We need to count 1-1 sequences only and to count them we use a notion of a permutation

Proof of Combinatorial Interpretation

Definition

A permutation of a set A is any function $f: A \overset{1-1,onto}{\longrightarrow} A$

Fact

For any non empty set set A of n elements the number of permutation of A is $n!$

Proof

By definition, we have so show that there are $n!$ functions $f: A \overset{1-1,onto}{\longrightarrow} A$. We carry the proof by **induction** over the number $n > 0$ of elements of the set A

Base Step Let $|A| = 1$. Hence $A = \{a\}$ and obviously there is only one function $f: \{a\} \overset{1-1,onto}{\longrightarrow} \{a\}$

By definition, $1! = 1$ and Base Step holds

Proof of Combinatorial Interpretation

InductIve Step Let $A = \{a_1, \ldots, a_n\}$ and $n > 1$

Assume that for any $B \subset A$, such that $|B| = n - 1$ there are $(n-1)!$ functions that map $f: B \xrightarrow{1-1, onto} B$

In order to **count** all functions

$$f: \{a_1, \ldots, a_n\} \xrightarrow{1-1, onto} \{a_1, \ldots, a_n\}$$

we divide them into n disjoint groups **G1, G2, . . . Gn** as follows

G1 consists of all functions f, such that

$$f(a_1) = a_1$$

By inductive assumption, **G1** contains $(n-1)!$ functions

Proof of Combinatorial Interpretation

G2 consists of all functions f, such that

$$f(a_2) = a_2$$

By inductive assumption, **G2** contains $(n-1)!$ functions

In general, **Gk** consists of all functions f, such that

$$f(a_k) = a_k$$

for $k = 1, 2, \ldots n$

By inductive assumption, each **Gk** contains $(n-1)!$ functions

Proof of Combinatorial Interpretation

We have divided the set of all functions into n disjoint groups, each containing $(n-1)!$ functions

Hence all together there are $n! = n(n-1)!$ functions

$$f : A \stackrel{1-1, onto}{\longrightarrow} A$$

This ends the proof of the **Fact** and we go back to the proof of the **Combinatorial Interpretation** as follows

Proof of Combinatorial Interpretation

Back to Step 1

Let $|A| = n$ be any n-element set

We **count** now all possible 1-1, k-element **sequences** out of elements of A as follows.

The 1-1, k-element **sequences** are of the form

$$b_1, b_2, \ldots, b_k \quad \text{for } b_i \neq b_j \text{ and } k \geq 1$$

1. $k = 1$

b_1 - there are n choices, for any $b_1 \in A$

2. $k = 2$

b_1, b_2 - there are n - 1 choices, for any $b_2 \in A - \{b_1\}$

Proof of Combinatorial Interpretation

3. $k = 3$

$b_1,\ b_2,\ b_3$ - there are n - 2 choices, for any $b_3 \in A - \{b_1, b_2\}$

Induction (really)

3. $k = i$

$b_1 b_2 \ldots . b_i$ - there are $(n - i + 1)$ choices for any $b_i \in A - \{b_1, b_2, \ldots, b_{i-1}\}$

All together we have $n(n-1)\ldots(n-k+1)$ possible 1-1 **sequences**

$$b_1,\ b_2, \ldots, b_k$$

Proof of Combinatorial Interpretation

Step 2

In **Combinatorial Interpretation** $\binom{n}{k}$ represents how many are there k-element **subsets** of the set A

We proved that there are $n(n-1)...(n-k+1)$ possible 1-1, k-element **sequences**

Now we have to establish a relationship between the 1-1 **sequences** $b_1, b_2, \dots, b_k$ and corresponding **subsets** $\{b_1, b_2, \dots, b_k\}$

Observation

Sets: $\{b_1, b_2, \dots, b_k\} = \{b_2, b_2, \dots, b_k\}$

Sequences : $b_1, b_2, \dots, b_k \neq b_2, b_2, \dots, b_k$

Proof of Combinatorial Interpretation

Different **sequences** $b_1, b_2, \ldots, b_k$ can represent the same **set** $\{b_1, b_2, \ldots, b_k\}$

Question: How many are there of all possible set representations $\{b_1, b_2, \ldots, b_k\}$ of the 1-1 **sequence** $b_1, b_2, \ldots, b_k$?

Answer: as many as **permutation** of the set $\{b_1, b_2, \ldots, b_k\}$, i.e. $k!$

Hence

$$\binom{n}{k} = \frac{\textit{number of sequences}}{k!} = \frac{n(n-1)...(n-k+1)}{k!}$$

This **ends** the proof

Generalization

We defined

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)\ \ldots\ 2\cdot 1}$$

i.e. by the formula

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

for $n, k \in N,\ k \geq 0,\ \ k \leq n$

We also proved the **Combinatorial Statement** that $\binom{n}{k}$ represents the **number** of ways to **choose** a k-element **subset** from an n-element **set.**

We defined

$$0! = 1 \quad \text{and} \quad x^{\underline{0}} = 1$$

Generalization

We generalize now the definition of $\binom{n}{k}$ as follows.

Consider a function $f: R \longrightarrow R$ given by a formula (for fixed $k \in Z$)

$$f(x) = x^{\underline{k}} = x(x-1)\ldots(x-k+1)$$

and $x^{\underline{0}} = 1$

or, more precisely, a function

$$f: R \times Z \longrightarrow R$$

given by formula

$$f(x,k) = \begin{cases} \frac{x^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Definition

Definition

For any $x \in R, \quad k \in Z$ we define

$$\binom{x}{k} = \begin{cases} \frac{x^k}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

BOOk uses notation $r \in R$ and defines

$$\binom{r}{k} = \begin{cases} \frac{r^k}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Examples

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} \quad \text{for} \quad k \geq 0,\ x \in R$$

$$x^{\underline{k}} = x(x-1)...(x-k+1)$$

We evaluate

$$\binom{-1}{3} = \frac{-1^{\underline{3}}}{3!} = \frac{(-1)(-2)(-3)}{1.2.3} = -1$$

$$\binom{-1}{-1} = 0 \ \text{ as } \ k < 0 \text{ and } \binom{1}{1} = 1$$

In general

$$\binom{n}{n} = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Examples

We evaluate

$$\binom{\sqrt{2}}{3} = \frac{\underline{\sqrt{2}}^{3}}{3!} = \frac{(\sqrt{2})(\sqrt{2}-1)(\sqrt{2}-2)}{1 \cdot 2 \cdot 3}$$

$$\underline{\sqrt{2}}^{3} = (\sqrt{2})(\sqrt{2}-1)(\sqrt{2}-2)$$

NO **Combinatorial Interpretation** HERE

Generalization

We defined

$$\binom{x}{k} = \begin{cases} \frac{x^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

for any $x \in R, \quad k \in Z$

Reminder

$$\binom{n}{n} = 1, \quad \text{for} \quad n \in N$$

$$\binom{n}{n} = 0, \quad \text{for} \quad n < 0$$

$$\binom{n}{k} = 0, \quad \text{for} \quad k > n, \quad k \geq 0$$

Symmetry Property

SP $\binom{n}{k} = \binom{n}{n-k}$ for any $n, k \in N, \quad 0 \leq k \leq n$

Proof We evaluate, by definition,

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$$

Combinatorial Interpretation

$\binom{n}{k}$ - - k chosen element from out of n

$\binom{n}{n-k}$ - - n-k unchosen element out of n

Symmetric Property

We proved that $\binom{n}{k} = \binom{n}{n-k}$ for $k, n \in N$ and $0 \leq k \leq n$

Case $k < 0$

We have $\binom{n}{k} = 0$ and

$\binom{n}{n-k} = \binom{n}{s} = 0$ as $s > n$

Case $k > n$

We have $\binom{n}{k} = 0$ and

$\binom{n}{n-k} = \binom{n}{s} = 0$ as $s < 0$

Symmetric Property Generalization

We have proved the

Symmetry Property Generalization

$$\textbf{SP} \quad \binom{n}{k} = \binom{n}{n-k}$$

holds for all $n \in N,\ k \in Z$

We will show now that it **can't be generalized** to $n \in Z$

Symmetric Property Generalization

For **example**, take $n = -1$ and any $k \geq 0$

We evaluate

$$\binom{-1}{k} = \frac{-1^{\underline{k}}}{k!} = \frac{(-1)(-2)...(-1-k+1)}{k!} = \frac{k!-1^{\underline{k}}}{k!} = (-1)^k$$

where $\quad x^{\underline{k}} = x(x-1)...(x-k+1)$

Now we evaluate

$$\binom{-1}{-1-k} = 0 \quad \text{for all} \quad k \geq 0$$

This proves that

$$\binom{-1}{k} \neq \binom{-1}{-1-k} \quad \text{for all} \quad k \geq 0$$

Absorption Identities

Absorption Identity

A1 $$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1} \quad \text{for} \quad x \in R, \quad k \in Z - 0$$

Proof We first proof that

$$x^{\underline{k}} = x(x-1)^{\underline{k-1}}$$

as follows

$$x(x-1)^{\underline{k-1}} = x(x-1)(x-2)\ldots((x-1)-(k-1)+1)$$

$$= x(x-1)\ldots((x-k+1) = x^{\underline{k}}$$

Absorption Identities

We evaluate now

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} = \frac{x(x-1)^{\underline{k-1}}}{k(k-1)!} = \frac{x}{k}\binom{x-1}{k-1}$$

This **ends** the proof.

We multiply both sides of the identity **A1** by k and get

$$\textbf{A2} \quad k\binom{x}{k} = x\binom{x-1}{k-1} \quad \text{for} \quad x \in R, \quad k \in Z$$

Absorption Identities

We are going to prove now the following

$$\textbf{A3} \quad (x-k)\binom{x}{k} = x\binom{x-1}{k} \quad \text{for} \quad x \in R, \;\; k \in Z$$

Proof We carry the proof in two stages

Stage 1: we prove **A3** for $x \in N, \; k \in Z$ using the Symmetry Property **SP**

$$\binom{n}{k} = \binom{n}{n-k}$$

that only holds for $x \in N$

Stage 2: we use a **Polynomial Argument** (to be defined) to extend Stage 1 case to $x \in N, \quad k \in Z$

Absorption Identities

Stage 1: we assume that $x \in N$ and evaluate

$$(x-k)\binom{x}{k} =^{\textbf{SP}} (x-k)\binom{x}{(x-k)}$$

$$= x\binom{x-1}{x-k-1} \text{ use A2 for } k := x-k$$

$$= x\binom{x-1}{(x-1)-k} =^{\textbf{SP}} x\binom{x-1}{k}$$

This **proves**

$$(x-k)\binom{x}{k} = x\binom{x-1}{k} \quad \text{for} \quad x \in N, \;\; k \in Z$$

Stage 2: Polynomial Argument

Observe the the equality

$$(x-k)\binom{x}{k} = x\binom{x-1}{k} \text{ for } \quad x \in R, \quad k \in Z$$

is an equality of the following two **polynomials** of the degree $(k+1)$ over R with integer coordinates

$$L(x) \ = \ (x-k)\binom{x}{k} \ = \ a_{k+1}x^{k+1} \ + \ \dots \ + a_0$$

$$P(x) \ = \ x\binom{x-1}{k} \ = b_{k+1}x^{k+1} \ + \ \dots \ + b_0$$

as

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} = \frac{x(x-1)...(x-k+1)}{k!}$$

is a **polynomial** of the degree k

Polynomial Argument

Polynomial Theorem 1

Let $w(x) = a_n x^n + \ldots + a_0$ be a polynomial of the degree n with $a_i \in Z, i = 0, \ldots, n$ and $n \neq 0$.

Then the equation $w(x) = 0$ has at most n solutions; i.e.

$$|\{x \in R : \; w(x) = 0\}| \leq n$$

Polynomial Theorem 2

Let $w(x) = a_n x^n + \ldots + a_0$ be a polynomial with of the degree n with $a_i \in Z, \; i = 0, \ldots, n$ and $n \neq 0$, such that

$$|\{x \in R : \; w(x) = 0\}| > n$$

Then

$$w(x) = 0 \quad \textbf{for all} \quad x \in R$$

Polynomial Argument

Back to Absorption Identity

$$\textbf{A3} \quad (x-k)\binom{x}{k} = x\binom{x-1}{k} \text{ for } \quad x \in R, \quad k \in Z$$

We write it as

$$L(x) = P(x), \quad \text{or} \quad L(x) - P(x) = 0, \quad \textbf{for all } x \in R,$$

where $L(x), P(x)$ are two **polynomials** of the degree $(k+1)$ over R, with integer coordinates

$$L(x) = (x-k)\binom{x}{k} = a_{k+1}x^{k+1} + \ldots + a_0$$

$$P(x) = x\binom{x-1}{k} = b_{k+1}x^{k+1} + \ldots + b_0$$

Polynomial Argument

Observe that we have just proved **A3** for all $x \in N$, i.e. we proved that

$$|\{x \in R: \; L(x) - P(x) = 0\}| = |N| = \aleph_0 \; > \; k \quad \text{for all } k \in Z$$

By **Polynomial Theorem 2**,

$$L(x) - P(x) = 0, \quad \textbf{for all } x \in R$$

and hence we have **proved** the Absorption Identity

$$\textbf{A3} \quad (x-k)\binom{x}{k} = x\binom{x-1}{k} \text{ for } \; x \in R, \; k \in Z$$

Absorption Identities

We are going to prove now yet another Absorption Identity

$$\textbf{A4} \quad \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1} \quad \text{for} \quad x \in R, \;\; k \in Z$$

We present here two proofs

Proof 1 We carry the proof in two stages

Stage 1: we prove **A4** for $x \in N, \; k \in Z$

Stage 2: we use a **Polynomial Argument** to extend Stage 1 case to $x \in N, \quad k \in Z$

Proof 2 We use Absorption Identities **A2** and **A3**- left as an exercise

Polynomial Argument

We prove the case $x \in N$ by a straightforward evaluation.

We use the **Polynomial Argument** as follows

Let

$$L(x) = \binom{x}{k}$$ - polynomial of the degree k

$$P(x) = \binom{x-1}{k} + \binom{x-1}{k-1}$$ - polynomial of the degree k

We proved that

$$L(x) - P(x) = 0, \quad \textbf{for all} \quad x \in N$$

Polynomial Argument

Hence

$$|\{x \in R: \; L(x) - P(x) = 0\}| = |N| = \aleph_0 > k \quad \text{for all } k \in Z$$

By **Polynomial Theorem 2**,

$$L(x) - P(x) = 0, \quad \textbf{for all } x \in R$$

and hence we have **proved** the

$$\textbf{A4} \quad \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1} \quad \text{for} \quad x \in R, \; k \in Z$$