cse547
DISCRETE MATHEMATICS

Professor Anita Wasilewska

LECTURE 2

CHAPTER 1
PART THREE: The Josephus Problem

Josephus Story

Flavius Josephus was a historian of 1st century

During Jewish-Roman war Josephus was among 41 Jewish rebels captured by the Romans

They preferred **suicide** to the **capture** and decided to form a circle and to **kill** every third person until **no one** was left

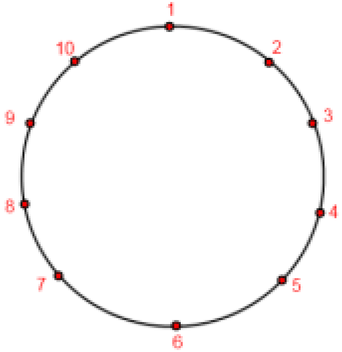
Josephus with with one friend wanted none of this **suicide** nonsense and he calculated **where** he and his friend should **stand** to avoid being killed and they were **saved**

The Josephus Problem - Our variation

n people around the **circle** and we **eliminate** each second remaining person **until** one survives

We denote by $J(n)$ the **position** of a surviver

Example $n = 10$



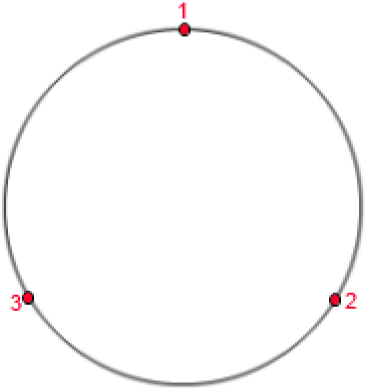
Elimination order: 2, 4, 6, 8, 10, 3, 7, 1, 9.

As a result, number 5 survives, i.e. $J(10) = 5$

Problem: Determine survivor number J(n)

We **evaluate** now J(n) for n=1,2, . . . 6

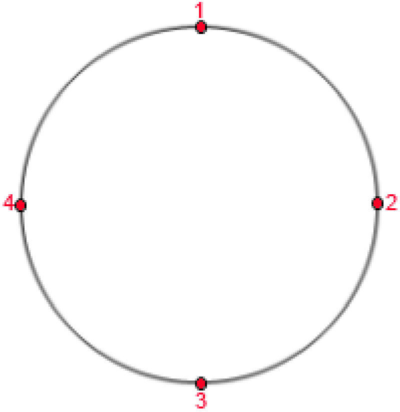
J(1)=1, J(2) = 1, J(3):



We get that J(3)=3

Determine survivor number J(n)

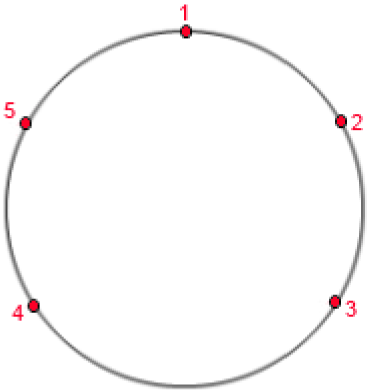
Picture for J(4):



We get J(4)=1

Determine survivor number J(n)

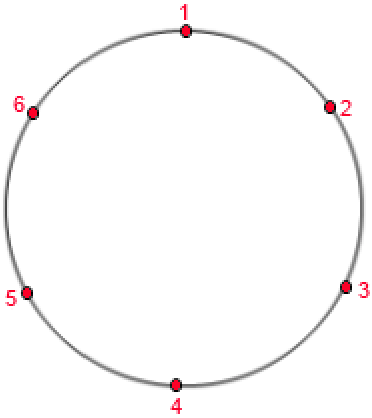
Picture for J(5):



We get J(5)=3

Problem: Determine survivor number J(n)

Picture for J(6):



We get J(6)=5

Determine survivor number J(n)

We put our results in a table:

n	1	2	3	4	5	6
J(n)	1	1	3	1	3	5

Observation

All our J(n) after the **first run** are odd numbers

Fact

First trip **eliminates** all even numbers

Determine survivor number J(n)

Fact

First trip **eliminates** all even numbers

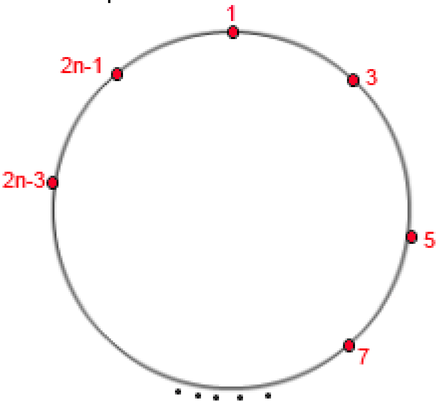
Observation

If $n \in \text{EVEN}$ we arrive to a similar situation we started with with half as many people (numbering has changed)

Determine survivor number J(n)

ASSUME that we START with 2n people

After first trip we have



3 goes out next

This is like starting with n except **each** person has been doubled and decreased by 1

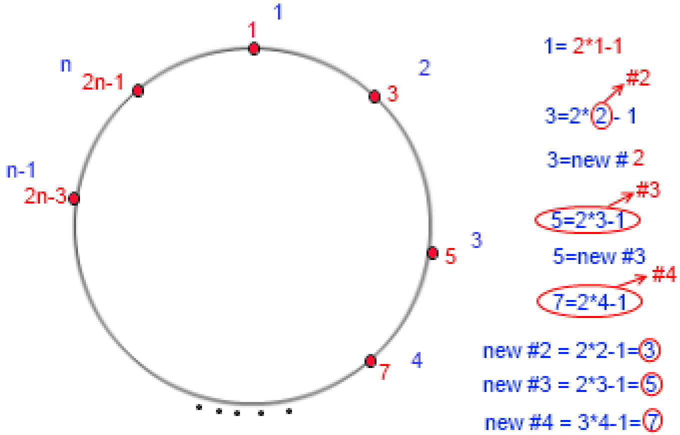
Determine survivor number J(n)

Case n=2n

We get **J(2n)=2J(n) -1** (each person has been doubled and decreased by 1)

We know that J(10)=5, so J(20) = 2J(10)-1 = 2*5-1 = 9

Re-numbering

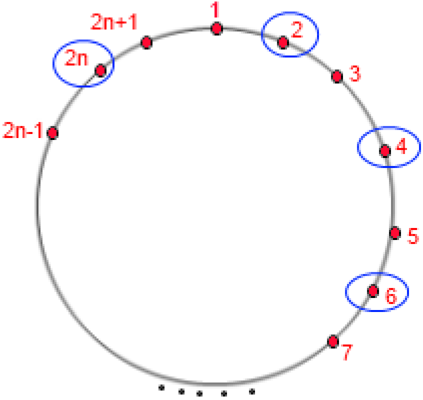


Determine survivor number J(n)

Case n=2n+1

ASSUME that we start with 2n+1 people:

First looks like that

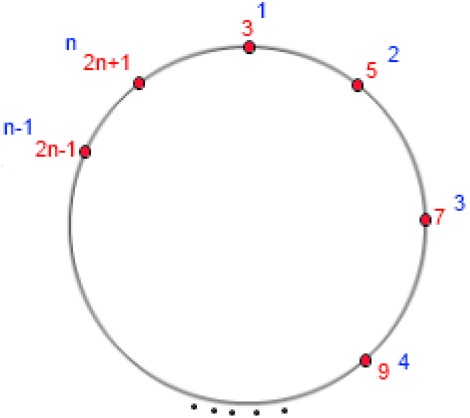


1 is wipped out after 2n

We want to have n-elements after **first** round

Determine survivor number J(n)

After the first trip we have

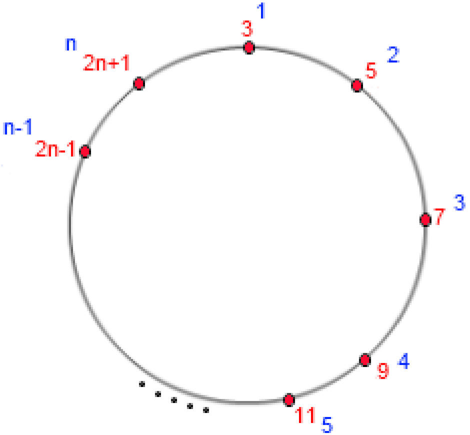


This is like starting with n except that now each person is doubled and increased by 1

Determine survivor number J(n)

CASE n=2n + 1 c.d.

Re-numbering



3=2*1-1
5=2*2+1
7=2*3+1

3=new #1 3=2*1+1
5=new #2 5=2*2+1

new 1 =2*1+1=3 → Old
new 2=2*2+1=5 → Old
new 3=2*3+1=7 → Old

Formula: new number k = 2k+1

J(2n+1)=new number J(n)

J(2n+1)=2J(n)+1

Recurrence Formula for J(n)

The Recurrence Formula RF for J(n) is:

$J(1) = 1$

$J(2n) = 2J(n) - 1$

$J(2n + 1) = 2J(n) + 1$

Remember that J(k) is a position of the **survivor**

This formula is more efficient then getting F(n) from F(n-1)

It reduces n by factor 2 each time it is applied

We need only 19 application to evaluate $J(10^6)$

From Recursive Formula to Closed Form Formula

In order to find a **Closed Form Formula** (CF) equivalent to given **Recursive Formula** RF we ALWAYS follow the the Steps 1 - 4 listed below.

- Step 1 Compute from recurrence RF a TABLE for some initial values. In our case RF is: $J(1) = 1$, $J(2n) = 2J(n) - 1$, $J(2n+1) = 2J(n) + 1$
- Step 2 **Look** for a pattern formed by the values in the TABLE
- Step 3 **Find** - guess a closed form formula CF for the pattern
- Step 4 **Prove** by Mathematical Induction that RF = CF

TABLE FOR J(n)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1
	G1	G2		G3				G4								G5

Observation: $J(n) = 1$ for $n = 2^k, \quad k = 0, 1, ..$

Next step: we form groups of J(n) for n consecutive powers of 2 and observe that

J(n)	G1	G2	G3	G4	G5	...
n	2^0	$2^1 + l$	$2^2 + l$	$2^3 + l$	$2^4 + l$	...

for $0 \leq l < 2^{(k-1)}$ and $k = 1, 2, ...5,$

Computation of J(n)

Observe that for each **group** G_k the corresponding n are $n = 2^{k-1} + l$ for all $0 \leq l < 2^{(k-1)}$ and the value of $J(n)$ for $n = 2^k + l$ i.e. $J(n) = J(2^k + l)$ **increases** by 2 within the **group**

Let's now make a TABLE for the group G3

J(n)	1	3	5	7 = 2l+1
n	2^2	$2^2 + l$	$2^2 + 2$	$2^2 + 3$
	l=0	l=1	l=2	l=3

Guess for CF formula for J(n)

Given $n = 2^{k-1} + l$ we **observed** that $J(n) = 2l + 1$

We **guess** that our CF formula is

$$J(2^k + l) = 2l + 1,$$

for any $k \geq 0,\ \ 0 \leq l < 2^k$

Representation of n

$n = 2^k + l$ is called a **representation** of n when

l is a **remainder** by dividing n by 2^k and

k is the largest power of 2 not exceeding n

Observe that $2^k \leq n < 2^{k+1}$, $l = n - 2^k$ and so

$0 \leq l < 2^{k+1} - 2^k = 2^m$, i.e.

$$0 \leq l < 2^k$$

Proof RF = CF

RF: $J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1$

CF: $J(2^k + l) = 2l + 1$, for $n = 2^k + l$, $k \geq 0$, $0 \leq l < 2^k$

Proof: by Mathematical Induction on k

Base Case: k=0.

Observe that $0 \geq l < 2^0 = 1$, and l = 0 , $n = 2^0 + 0 = 1$, i.e. $n = 1$.

We evaluate $J(1) = 1$, $J(2^0) = 1$, i.e.

$$RF = CF$$

Proof RF = CF

Induction Step over k has two cases

c1: **n** $\in$ **even** and $J(2n) = 2J(n) - 1$

c2: **n** $\in$ **odd** and $J(2n+1) = 2J(n) + 1$

Induction Assumption for k is

$J(2^{k-1} + l) = 2l + 1$, for $0 \leq l < 2^{k-1}$

case c1: **n** $\in$ **even**

put n:= 2n, i.e. $2^k + l = 2n$, $0 \leq l < 2^k$

Observe that

$2^k + l = 2n$ iff $l \in$ even, i.e. $l = 2m$, and

l/2 = m $\in N$ and $0 \leq \frac{l}{2} < 2^{k-1}$.

Proof RF = CF

We evaluate n from $2^k + l = 2n$ as follows

$n = \frac{2^k + l}{2}$,

$n = 2^{k-1} + \frac{l}{2}$, for $0 \leq \frac{l}{2} < 2^{k-1}$, $\frac{l}{2} \in N$

Proof in case **c1: n** $\in$ **even** and $J(2n) = 2J(n) - 1$

Reminder: CF: $J(2^k + l) = 2l + 1$ for $n = 2^k + l$

$J(2^k + l) =^{reprn} 2J(2^{k-1} + \frac{l}{2}) - 1$

$=^{ind} 2(2\frac{l}{2} + 1) - 1 = 2l + 2 - 1$

$= 2l + 1$

Proof RF = CF

Proof in case **c2:** **n** $\in$ **odd** and $J(2n+1) = 2J(n)+1$

Inductive Assumption: $J(2^{k-1}+l)$ = $2l+1$, for $0 \leq l < 2^{k-1}$

Inductive Thesis: $J(2^k+l)$ = $2l+1$, for $0 \leq l < 2^k$

We put $n := 2n+1$ and observe that

$2^k + l = 2n+1$ iff l $\in$ odd, i.e.

l = 2m+1, for certain $m \in N$, l -1 = 2m, and $\frac{l-1}{2} = m \in N$

Proof of RF = CF

Let $J(2n+1) = 2J(n)+1$

We evaluate, as before n from $2^k + l = 2n+1$

$2^k + l - 1 = 2n$ and we get the representation of n

$n = 2^{k-1} + \frac{l-1}{2}$

Reminder: CF: $J(2^k + l) = 2l + 1$ for $n = 2^k + l$

Proof RF = CF in case **c2: n** $\in$ **odd** and

$J(2n+1) = 2J(n)+1$ is now as follows

$J(2^k + l) =^{reprn} 2J(2^{k-1} + \frac{l-1}{2}) + 1$

$=^{ind} 2(2\frac{l-1}{2} + 1) + 1 = 2(l - 1 + 1) + 1$

$= 2l + 1$

Some Facts

Fact 1 $\forall_m \; J(2^m) = 1$

Proof by induction over m

Observe that $2^m \in \mathit{even}$, so we use the formula

$J(2n) = 2J(n) - 1$, and get

$$J(2^m) = J(2 * 2^{m-1}) \;=^{Jdef}\; 2J(2^{m-1}) - 1 =^{ind} 2 * 1 - 1 = 1$$

Hence we also have

Fact 2

First person will always **survive** whenever n is a power of 2

General Case

Fact 3

Let $n = 2^m + l$

The first remaining person, the **survivor** is number $2l + 1$

Our solution for the **proof**

Observe that the number of people is **reduced** to power of 2 after there have been l **executions**

$$J(2^m + l) = 2l + 1$$

where $n = 2^m + l$ and $0 \leq l < 2^m$ depends heavily on powers of 2

Let's look now at the binary expansion of n and see how we can **simplify** the computations

Binary Expansion of n

Definition

$$n = (b_m b_{m-1} ... b_1 b_0)_2$$

stands for

$$n = b_m 2^m + b_{m-1} 2^{m-1} + .. b_1 2 + b_0$$

for

$$b_i \in 0, 1, \quad b_m = 1$$

Binary Expansion of n

EXAMPLE: n=100

n = $(1\ 1\ 0\ 0\ 1\ 0\ 0)_2$

$2^6 2^5 2^4 2^3 2^2 2^1 2^0$

n = $2^6 + 2^5 + 2^2 = 64 + 32 + 4 + 100$

Binary Expansion of n

Let now :

n = $2^m + l$, $\quad 0 \leq l < 2^m$

we have the following binary expansions:

1) $l = (0, b_{m-1}, .., b_1, b_0)_2$ as $l < 2^m$

2) $2l = (b_{m-1}, .., b_1, b_0, 0)_2$ as

$$l = b_{m-1}2^{m-1} + .. + b_1 2 + b_0$$
$$2l = b_{m-1}2^m + .. + b_1 2^2 + b_0 2 + 0$$

3) $2^m = (1, 0, ..., 0)_2, \quad 1 = (0...1)_2$

4) $n = 2^m + l$

$n = (1, b_{m-1}, .., b_1, b_0)_2$ from 1 + 3

5) $2l + 1 = (b_{m-1}, b_{m-2}, .., b_0, 1)_2$ from $2 + 3$

Binary Expansion Josephus

Consider now a **closed formula**

$CF: \quad J(n) = 2l + 1, \quad \text{for} \quad n = 2^m + l$

We use

5) $2l + 1 = (b_{m-1}, b_{m-2}, .., b_0, 1)_2$

and re-write the **closed** formula CF as a **binary expansion** formula BF as follows

$BF: \quad J((b_m, b_{m-1}, .., b_1, b_0)_2) = (b_{m-1}, .., b_1, b_0, b_m)_2$

because $b_m = 1$ in the binary expansion of n, we get

$$BF: \quad J((1, b_{m-1}, .., b_1, b_0)_2) = (b_{m-1}, .., b_1, b_0, 1)_2$$

Binary Expansion Josephus

Example: Find J(100)

$n = 100 = (1100100)_2$

$$J(100) = J((1100100)_2) =^{BF} (1001001)_2$$

J(100) = 64 + 8 + 1 = 73

$$BF: \quad J((1, b_{m-1}, .., b_1, b_0)_2) = ((b_{m-1}, .., b_1, b_0, 1)_2$$

Josephus Generalization

Our function $J: N - \{0\} \longrightarrow N$ is defined as

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

We generalize it to function $f: N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \quad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \quad n \geq 1$$

Observe that J = f for $\alpha = 1, \beta = -1, \gamma = 1$

NEXT STEP: Find a **Closed** Formula for f