Instruction for Assignment Two: Differential Topology

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Differential Topology: Theoretic Proofs

Brouwer Fixed Point

Problem (Mobius Transformation)

Let $\mathbb{D} = \{z \in \mathbb{C} || z| \leq 1\}$ *be the unit planar disk. A Möbius transformation* $\varphi : \mathbb{D} \to \mathbb{D}$ *is given by*

$$\varphi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

According to Brouwer fixed point theorem, there should be fixed points. Compute the fixed points, what are the norms of them ?

Surface Fixed Points

Problem

By stereo-projection, the unit sphere $\mathbb{S}^2$ is mapped onto the complex plane $\mathbb{C} \cup \{\infty\}$, a Möbius transformation $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is given by

$$\varphi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},$$

1. *Show the mapping is homotopic to the identity map.*
2. *How many fixed points are there ?*
3. *What are the indices of each of them ?*

Cohomology Group

Problem (Cohomology Group)

In our simplicial cohomology group basis $H^1(M, \mathbb{R})$ algorithm, we compute the 1-forms $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$, show that

1. *each ω_k is closed by not exact;*
2. *they are linearly independent, so form a basis of $H^1(M, \mathbb{R})$*

Harmonic Differential

Problem

Suppose a complex function $f : \mathbb{C} \to \mathbb{C}$ *is holomorphic,*
$f : x + iy \mapsto u + iv$,

1. *Write down the Cauchy-Riemann equation for* f;
2. *Show both* du *and* dv *are harmonic 1-forms;*
3. *Show* $dv = *du$, *where* $*$ *is the Hodge star operator;*
4. *Show the mapping is angle-preserving.*

Harmonic Differential

Problem

Suppose $(S, \mathbf{g})$ *is a closed oriented surface with genus* g, (u, v) *is an isothermal parameter such that* $\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2)$

1. *Suppose* $\mathbf{v} \in T_pS$ *is a tangent vector, represented as* $\mathbf{v} = v_1\partial u + v_2\partial v$; $\omega \in T_p^*S$ *is a differential 1-form,* $\omega = \omega_1 du + \omega_2 dv$, *show that there is a vector field* $\mathbf{w}$ *corresponding to* ω, *such that for any tangent vector*
$$\omega(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{g}}.$$
What is the representation of $\mathbf{w}$?
2. *Show that* ω *must have zeros, where* $\omega_1(p) = \omega_2(p) = 0$.
3. *If* ω *is harmonic, are the positions of the zeros random? How to describe the zeros ?*