Instruction for Assignment Three: Differential Geometry

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Differential Geometry: Theoretic Proofs

Random Walk

Problem (Random Walk)

Suppose M is a genus zero triangulated polyhedral surface with a single boundary. A particle randomly walks along the edges, each step the particle reaches a vertex. We use ω_k to represent the vertex the particle reaches at the k-th step. Suppose at the k-th step, the particle is at an interior vertex v_i, then at the $k+1$-th step, the probability of the particle reaches an adjacent vertex v_j is:

$$Prob(\omega_{k+1} = v_j | \omega_k = v_i) = \frac{w_{ij}}{\sum_{v_i \sim v_k} w_{ik}}$$

where w_{ij} is the cotangent edge weight. The random walks terminates when the particle hits a boundary vertex.

Random Walk

Problem (Random Walk)

Fix a boundary vertex $v_k \in \partial M$, for any interior vertex v_i, a random walk ω starting from v_i, and the probability of ω first hits the boundary at v_k is denoted as $f(v_i)$.

1. *Prove that $f : M \to \mathbb{R}$ is a discrete harmonic function.*
2. *Suppose we want to solve a discrete Laplace equation with Dirichlet boundary condition, $h : M \to \mathbb{R}$,*

$$\begin{cases} \Delta h & = & 0 \\ h|_{\partial M} & = & g, \end{cases}$$

   design an algorithm based on random walks to solve it.

Gaussian Curvature

Problem (Gaussian Curvature)

Suppose S is a closed, orientable, compact C^2 surface embedded in $\mathbb{R}^3$,

1. *Show there is at least one point p, such that the Gaussian curvature at p is strictly positive;*
2. *Show that the surface S with a different Riemannian metric $\mathbf{g}$), such that the curvature induced by $\mathbf{g}$ is negative everywhere, can not be isometrically embedded in $\mathbb{R}^3$.*

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Problem (Special Atlas)

Suppose S *is a genus* $g > 1$ *closed, orientable, compact surface,* $\mathcal{A} = \{(U_i, \varphi_i)\}$ *is an atlas of the surface such that all the chart transition functions*

$$\varphi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are planar rigid motions. Does that kind of atlas exist or not ? Justify.

Cross Field

Problem (Cross Field)

Suppose $(S, \mathbf{g})$ is a closed genus one surface, $p, q \in S$ are distinct points on the surface, $\mathbf{g}$ is a flat metric with cone singularities at p and q, such that the discrete Gaussian curvature is $\frac{\pi}{2}$ at p and $-\frac{\pi}{2}$ at q. A cross field is to assocate a non-oriented frame (a cross) at each point. Is there a smooth cross field on the surface ? If such kind of field exists, how to construct it ?