Instruction for Assignment Four: Conformal Structure

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July 28, 2022

Conformal Structure: Theoretic Proofs

Teichmüller Space

Problem (Torus)

Suppose S is a topological torus, two Riemannian metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ on S are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:

$$\mathbf{g}_1(p) = e^{2\lambda(p)}\mathbf{g}_1(p),$$

the space of conformal equivalence classes of Riemannian metrics is called the Tecihüller space. What is the dimension of the Teichmüller space of a torus?

Teichmüller Space

Problem (Topological Quadrilateral)

Suppose S is a topological disk with a single boundary, the boundary is piecewise analytic curve. There are four boundary points on ∂S, p_0, p_1, p_2, p_3.

- *Show that there is a conformal map from S to a planar rectangle, the four points p_0, p_1, p_2, p_3 are mapped to the corners of the rectangle.*
- *Two Riemannian metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ on S are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:*

  $$\mathbf{g}_1(p) = e^{2\lambda(p)}\mathbf{g}_1(p),$$

  the space of conformal equivalence classes of Riemannian metrics is called the Tecihüller space. What is the dimension of the Teichmüller space of a topological quadrilateral?

Problem (Poly-Annulus)

Suppose S is a topological poly-annulus, two Riemannian metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ on S are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:

$$\mathbf{g}_1(p) = e^{2\lambda(p)}\mathbf{g}_1(p),$$

the space of conformal equivalence classes of Riemannian metrics is called the Tecihüller space. What is the dimension of the Teichmüller space of a poly-annulus?

Teichmüller Space

Problem (Möbius Transformation)

The hyperbolic space is the unit disk $|z| < 1$ *with a metric*

$$ds^2 = \frac{dzd\bar{z}}{(1 - z\bar{z})^2}.$$

A Möbius transformation has the form

$$z \mapsto e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Show that this type of M"obius transformation is hyperbolic isometric.

Teichmüller Space

Problem (High Genus Surface)

Suppose S is a high genus surface with a hyperbolic metric, its universal covering space is the hyperbolic plane, the Deck transformations are Möbius transformations.

- *Suppose $\{a_1, b_1, \cdots, a_g, b_g\}$ is a set of canonical fundamental group genrators of $\pi_1(S, p)$, the corresponding Deck transformation group generators are $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$. What are the relations among the fundamental group generators? What are the relations among the Deck transformation generators ?*
- *How many parameters do we need to describe the Deck transformation group generators ?*
- *All possible hyperbolic metrics on S form a space. What is the dimension of this space?*