Assignment Seven: Persistent Homology for Handle and Tunnel Loops

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

August 9, 2022

Tunnel Loops

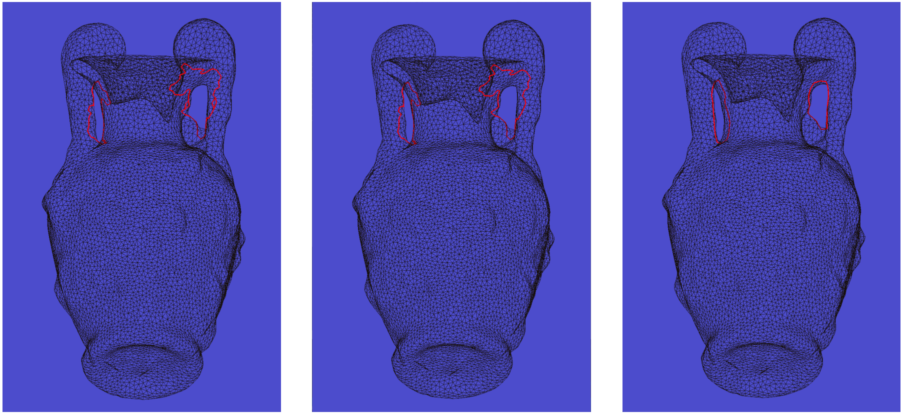


Figure: Handle and tunnel loops of the amphora model.

Tunnel Loops

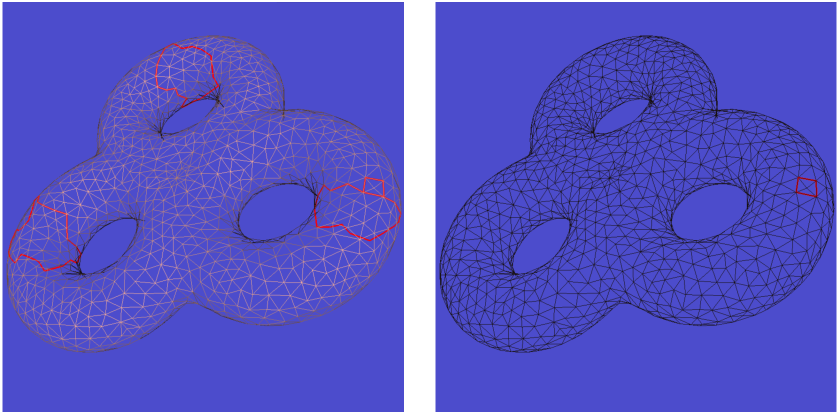


Figure: Null homotopy detection.

Tunnel Loops

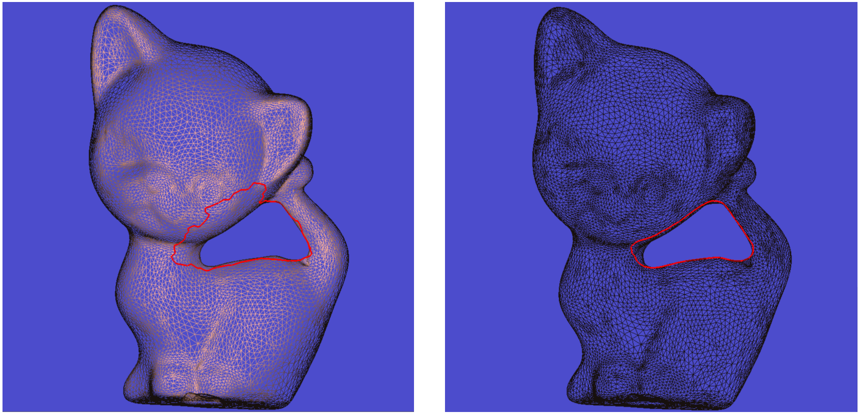


Figure: Birkhoff curve shortening.

Topological Torus

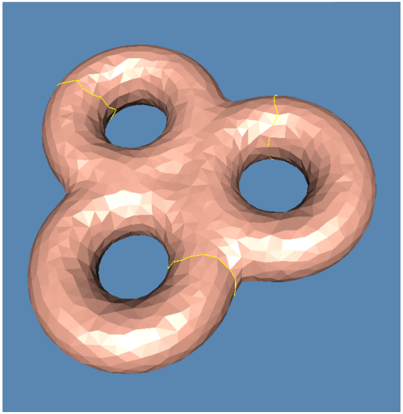
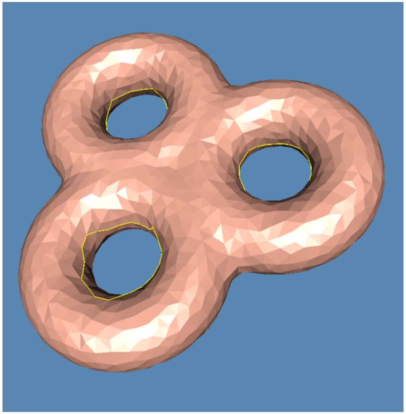



Figure: Handle and tunnel loops.

Topological Torus

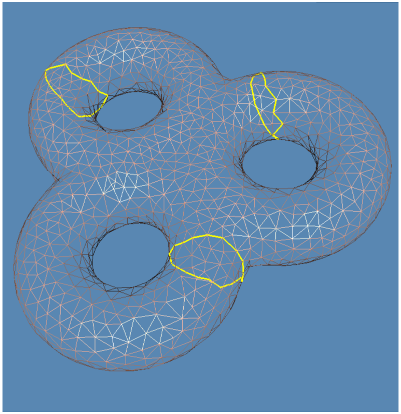
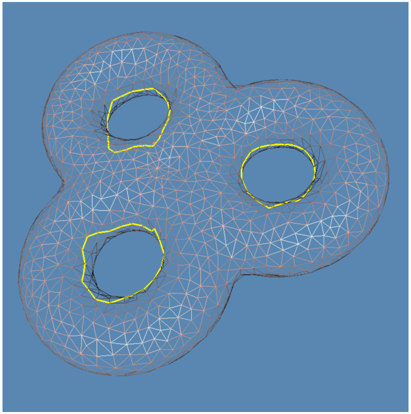



Figure: Handle and tunnel loops.

Topological Torus

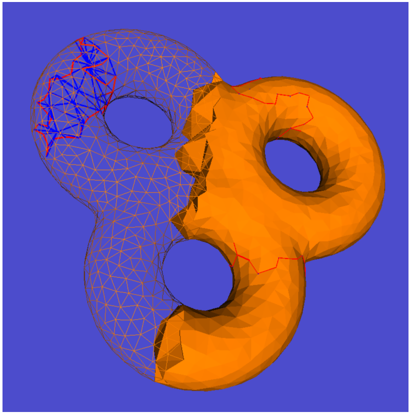
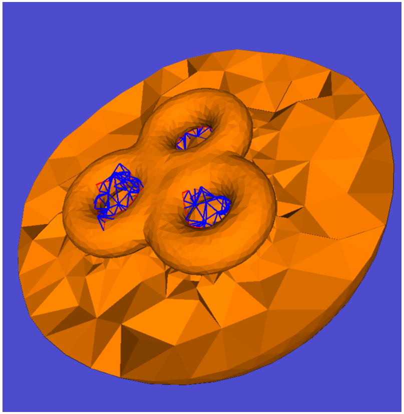



Figure: Interior and exterior volumes.

Volumetric Mesh Generation

The input is an oriented closed triangular mesh, use Dr. Hang Si's tetgen to generate the inteior and exterior volumetric mesh.

Interior Volume

Use Tetgen to generate the interior tetrahedral mesh inside the mesh M, denoted as I_M.

Exterior Volume

Construct a sphere enclosing the input mesh M, use Tetgen to generate a tetrahedral mesh between the sphere and the mesh M. Add the infinity point ∞, connect ∞ with each triangle face on the sphere to form a tetrahedron, denoted as O_M.

Filtration Generation

Interior Volume

Extract the boundary surface of the interior volume $M = \partial I_M$; Sort all the vertices, edges, faces of M,

$$\sigma_0^1, \sigma_0^2, \cdots, \sigma_0^{n_0}, \sigma_1^1, \sigma_1^2, \cdots, \sigma_1^{n_1}, \sigma_2^1, \sigma_2^2, \cdots, \sigma_2^{n_2}.$$

After that insert the interior vertices, edges, faces and tetrahedra of $I_M \setminus M$,

$$\tau_0^1, \tau_0^2, \cdots, \tau_0^{m_0}, \tau_1^1, \tau_1^2, \cdots, \tau_1^{m_1}, \tau_2^1, \tau_2^2, \cdots, \tau_2^{m_2}, \tau_3^1, \tau_3^2, \cdots, \tau_3^{m_3}.$$

Filtration Generation

Exterior Volume

Extract the boundary surface of the exterior volume $M = \partial O_M$; Sort all the vertices, edges, faces of M,

$$\sigma_0^1, \sigma_0^2, \cdots, \sigma_0^{n_0}, \sigma_1^1, \sigma_1^2, \cdots, \sigma_1^{n_1}, \sigma_2^1, \sigma_2^2, \cdots, \sigma_2^{n_2}.$$

After that insert the interior vertices, edges, faces and tetrahedra of $O_M \setminus M$,

$$\tau_0^1, \tau_0^2, \cdots, \tau_0^{m_0}, \tau_1^1, \tau_1^2, \cdots, \tau_1^{m_1}, \tau_2^1, \tau_2^2, \cdots, \tau_2^{m_2}, \tau_3^1, \tau_3^2, \cdots, \tau_3^{m_3}.$$

Pair Algorithm

Pair(σ)

```
1  $c = \partial_p \sigma$
2  $\tau$ is the youngest positive $(p-1)$-simplex in $c$.
3  **while** $\tau$ is paired and $c$ is not empty **do**
4      find $(\tau, d)$, $d$ is the $p$-simplex paired with $\tau$;
5      $c \leftarrow \partial_p d + c$
6      Update $\tau$ to be the youngest positive $(p-1)$-simplex in $c$
7  **end while**
8  **if** $c$ is not empty **then**
9      $\sigma$ is negative $p$-simplex and paired with $\tau$
10 **else**
11     $\sigma$ is a positive $p$-simplex
12 **endif**
```

Handle Loop and Tunnel Loop

1. The simplices on the surface M are added into the filtration in any arbitrary order. Since $H_1(M)$ is of rank $2g$, the algorithm Pair generates $2g$ number of unpaired positive edges.
2. The simplices up to dimension 2 in I are added into the filtration. Since $H_1(I)$ of rank g, half of $2g$ positive edges generated in step 1 get paired with the negative triangles in I. Each pair correponds to a killed loop, these g loops are handle loops.
3. Or the simplices up to dimension 2 in O are added into the filtration. Since $H_1(O)$ of rank g, half of $2g$ positive edges genrated in step 2 get paried with the negative triangles in O. Each pair corresponds to a killed loop, these g loops are tunnel loops.

Mark Loop Algorithm

Input: Trace Pair(τ, σ), τ is a handle loop generator, σ is its killer.
Output: a handle loop cycle c, the youngest generator of c is τ

```
1  $c = \partial_2 \sigma$
2  $\tau$ is the youngest generator edge in $c$.
3  while $\tau$ is paired and $c$ is not empty do
4      find $(\tau, d)$, $d$ is the killer face paired with $\tau$;
5      $c \leftarrow \partial_p d + c$
6      Update $\tau$ to be the youngest generator edge in $c$
7  if $\tau$ is on the boundary surface, then
8      break;
9  endif
10 end while
11 return the resultant $c$.
```

Mark Loop Algorithm

Lemma (Termination Condition)

In the mark loop algorithm, when τ is on the boundary surface, then the whole cycle c is also contained in the surface.

Proof.

1. By assumption, the generator τ corresponds to a handle loop L; the boundary of the killer $C = \partial\sigma$ is homologous to L, C is contained in the volume.
2. At each step in the while-loop, the transformation C preserves homologous class. Assume at the k-th step

$$C_k = L + \partial D_{S,k} + \partial D_{V,k},$$

where $D_{S,k}$ is a 2-chain on the surface; $D_{V,k}$ is a 2-chain in the volume, not on the surface. Then $\partial D_{V,k}$ is a cycle in the volume. □

Mark Loop Algorithm

Proof.

3. In the filtration, there is a step n, when all the edges of the volume have been added, but no face has been inserted to the complex $\mathbb{K}_n$. Then cycle $\partial D_{V,k}$ is in $\mathbb{K}_n$ and homological non-trivial. According to the key lemma, the youngest generator edge τ' of $\partial D_{V,k}$ is a cycle in the volume, younger than any cycle on the surface, hence $\tau' > \tau$.
4. Therefore, when τ becomes the youngest generator edge, $D_{V,k}$ must be empty,

$$C_k = L + \partial D_{S,k} \subset S$$

contained in the surface. □

Exact Component

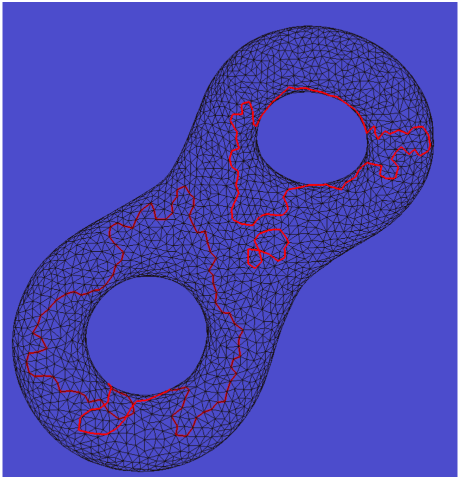


Figure: Step 1: the output of the handle loop algorithm.

Exact Component

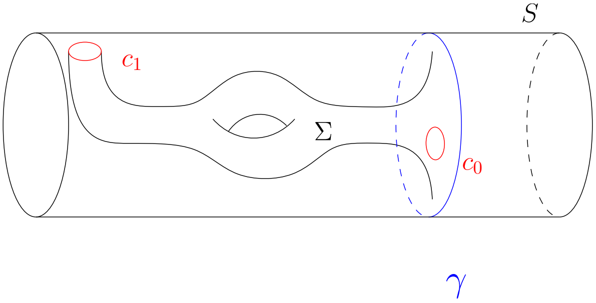


Figure: Exact component in the output of the mark loop algorithm. γ is the handle loop, c_0 is the boundary of the killer σ.

$$c_0 = \partial\sigma = c_1 + \gamma + \partial\Sigma, \quad [c_0] = [\gamma].$$

Exact Component

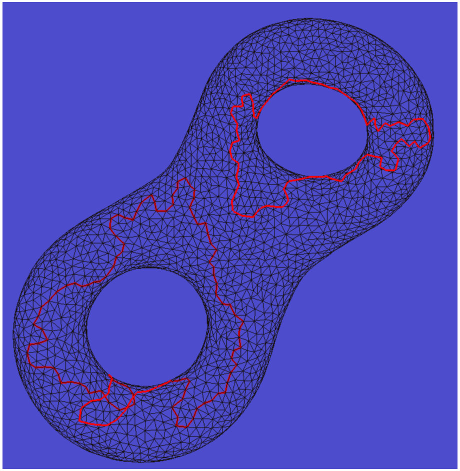


Figure: Step 2: Find the null homologous component and remove it.

Null Homological Cycle Detection

Input : a graph G on the mesh M labeled as sharp edges;
Output: remove null homological cycles

1. Build a spanning tree T of G, $G \setminus T = \{e_1, e_2, \ldots, e_k\}$;
2. Construct cycles $c_i = T \cup e_i$, $i = 1, 2, \cdots, k$;
3. Compute the persistent homology of the mesh M;
4. for each cycle c_i find the unpaired youngest generator; if one can not find the generator, then c_i is null homologous.

Exact Component

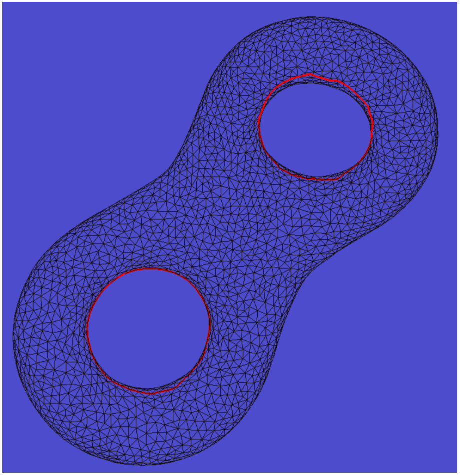
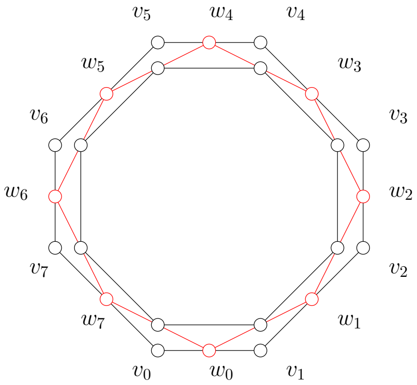
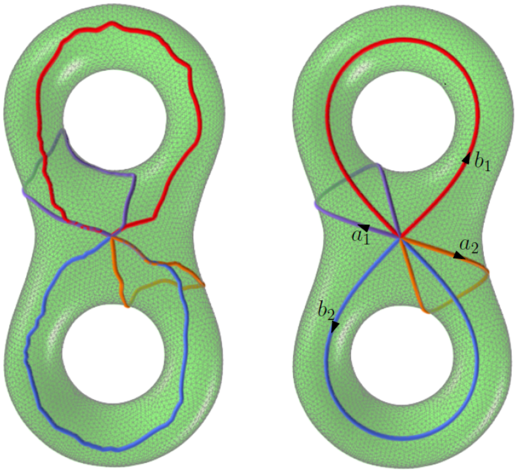


Figure: Step 3: Birkhoff curve shortening result.

Birkhoff curve shortening



Birkhoff curve shortening



Birkhoff curve shortening

Input : a loop c on M labeled as sharp edges;
Output: a shortened cycle homotopic to c;

1. Sort the vertices of c as $v_0, v_1, \dots, v_{n-1}$;
2. Find the shortest path between v_0 and $v_{n/3}$, and replace the sequence of edges between v_0 and $v_{n/3}$;
3. Find the shortest path between $v_{n/3}$ and $v_{2n/3}$, and replace the sequence of edges between $v_{n/3}$ and $v_{2n/3}$;
4. Find the shortest path between $v_{2n/3}$ and v_0, and replace the sequence of edges between $v_{2n/3}$ and v_0;
5. Cyclically shift the vertex sequence, and repeat step 2 through step 4.

Instruction

Dependencies

1. ‘DartLib‘, a volumetric mesh library based on Dart data structure.
2. ‘freeglut‘, a free-software/open-source alternative to the OpenGL Utility Toolkit (GLUT) library.

Directory Structure

- 3rdparty/DartLib, header files for volumetric mesh;
- HandleTunnelLoop/include, the header files for handle-tunnel loop computation;
- data,Some data models and batch scripts;
- CMakeLists.txt, CMake configuration file;
- resources, snapshot for circular slit mapping results;

Configuration

Before you start, read README.md carefully, then go three the following procedures, step by step.

1. Install [CMake](https://cmake.org/download/).
2. Download the source code of the C++ framework.
3. Configure and generate the project for Visual Studio.
4. Open the .sln using Visual Studio, and complie the solution.
5. Finish your code in your IDE.
6. Run the executable program.

Configure and generate the project

1. open a command window
2. cd Assignment_7_skeleton
3. mkdir build
4. cd build
5. cmake ..
6. open CCGHomework.sln inside the build directory.

Finish your code in your IDE

- You need to modify the file: HandleTunnelLoop.cpp;
- search for comments "insert your code"
- Modify functions:
  1. $CHandleTunnelLoop :: \_pair(std :: set < M :: CVertex* > \&vertices)$
  2. $CHandleTunnelLoop :: \_pair(std :: set < M :: CEdge* > \&edges)$
  3. $CHandleTunnelLoop :: \_pair(std :: set < M :: CFace* > \&faces)$
  4. $CHandleTunnelLoop :: \_mark\_loop(M :: CFace * killer)$

Finish your code in your IDE

Modify assignment one, CutGraph, to implement the algorithms for null homologous cycle detection and Birkhoff curve shortening.