# Assignment Eight: Surface Hyperbolic Structure

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# Canonical Fundamental Group Generator

## Step 1

Compute a set of canonical fundamental group generators of $S$,

$$\pi_1(S,p) = \langle a_1, b_1, \cdots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Based on Assignment 7 to compute handle loops and tunnel loops.

# Canonical Fundamental Group Generator

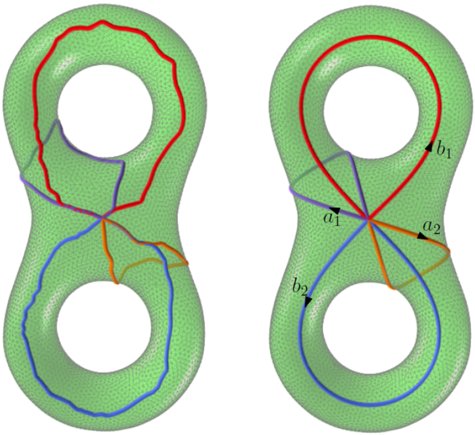


Figure: A set of canonical fundamental group generators.

# Hyperbolic Ricci Flow

## Step 2

Use hyperbolic Ricci flow to compute the uniformization metric.

1. Set the target curvature $\bar{K}$ to zeros everywhere;
2. set the conforaml factor $u$ to zeros for all vertices;
3. set the edge length
$$l_{ij} \leftarrow e^{\frac{u_i}{2}} y_{ij} e^{\frac{u_j}{2}}$$
4. Use hyperbolic cosine law to compute corner angles $\theta_{ij}^k$
5. Compute the vertex curvature $K_i$
6. Compute the gradient of the entropy energy $\nabla E = (\bar{K}_i - K_i)$
7. Compute the Hessian matrix of the entropy energy $H$
8. Solve linear system $H\delta u = \nabla E$
9. $\mathbf{u} \leftarrow \mathbf{u} + \lambda\delta\mathbf{u}$
10. Repeat step 3 through 9, until $\|\nabla E\| < \varepsilon$.

# Hyperbolic Isometric Embedding

## Step 3

1. Slice the mesh along the canonical fundamental group generators to get a fundamentable domain $\bar{S}$;
2. isometrically embed a face $f_0$ onto $\mathbb{H}^2$
3. enqueue the face $f_0$ to the queue $Q$, set $f_0$ as processed
4. **while** queue is not empty
5. $f_0 \leftarrow$ Pop $Q$
6. **for** each face $f$ adjacent to $f_0$ and unprocessed, embed it on $\mathbb{H}^2$
7. suppose $f = [v_0, v_1, v_2]$, $v_0$ and $v_1$ has been embedded, $\varphi(v_2)$ is the intersection of two hyperbolic circles, $c(v_0, l_{02})$ and $c(v_1, l_{12})$, and the orientation is counter-clockwise
8. enqueue $f$ to the queue $Q$
9. **endfor**
10. **while**

# Funchsian Group Generator

## Step 4

1. Locate the segments of $\varphi(\bar{S})$ to $a_k, b_k, a_k^{-1}, b_k^{-1}$, $k = 1, 2, \cdots, g$;
2. choose a pair of segments, $\gamma$, and $\gamma^{-1}$,
3. Find a Möbius transformation $\alpha_k$, such that

$$\alpha_k(b_k(0)) = b_k^{-1}(1), \quad \alpha_k(b_k(1)) = b_k^{-1}(0).$$

   and

$$\beta_k(b_k^{-1}(0)) = b_k(1), \quad \beta_k(b_k^{-1}(1)) = b_k(0).$$

4. Output the Fuchsian group genrators $\{\alpha_k, \beta_k\}_{k=1}^{g}$.

# Fuchsian Group Generator

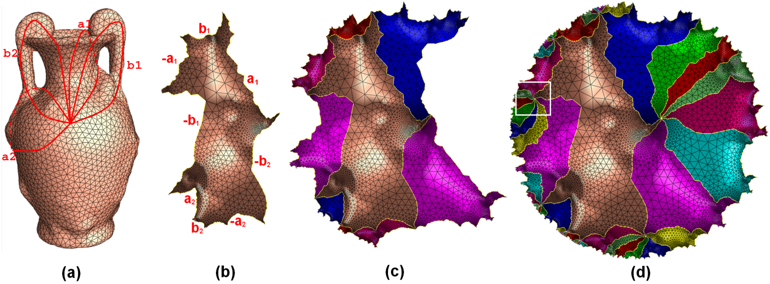


Figure: Fuchsian group generators.

# Hyperbolic Geodesic Boundary

## Step 5

1. Use the Fuchsian group generators to compute the Fuchsian transformations
2. Transform the embedding image of the fundamental domain to tessellate the hyperbolic disk
3. Replace each boundary segment of the fundamental domain by the unique hyperbolic geodesic
4. Recompute the fundamental domain and its embedding
5. Generator a finite portion of the universal covering space

# Hyperbolic Geodesic Boundary

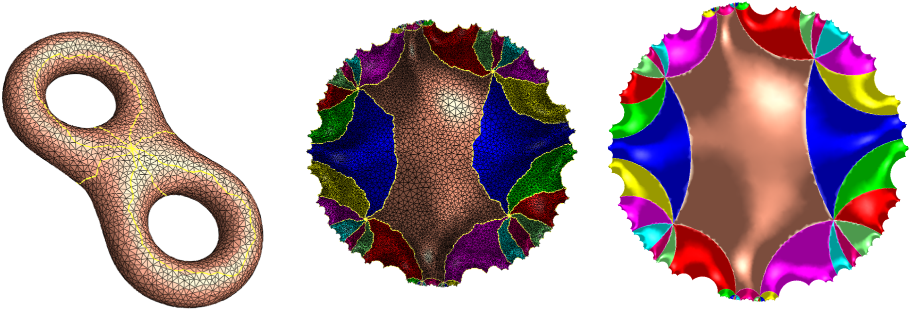

Figure: Fuchsian group generators.

# Hyperbolic Triangle

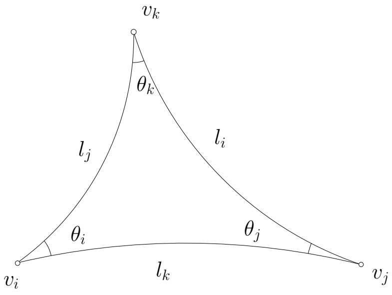


Figure: Hyperbolic triangle.

Cosine law:

$$\cos\theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

Sine law:

$$\frac{\sinh l_i}{\sin\theta_i} = \frac{\sinh l_j}{\sin\theta_j} = \frac{\sinh l_k}{\sin\theta_k}$$

Area

$$A = \frac{1}{2}\sinh l_j \sinh l_k \sin\theta_i$$

# Curvature

## Definition (Discrete Curvature)

Given a discrete surface with hyperbolic background geometry $(S, V, \mathcal{T}, l)$, every triangle is a hyperbolic geodesic triangle, the vertex discrete curvature is defined as the angle deficit

$$K(v) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk}, & v \notin \partial S \\ \pi - \sum_{jk} \theta_i^{jk}, & v \in \partial S \end{cases}$$

## Theorem (Gauss-Bonnet)

*The discrete Gauss-Bonnet theorem is represented as:*

$$\sum_{v \notin \partial S} K(v) + \sum_{v \in \partial S} K(v) - Area(S) = 2\pi\chi(S)$$

# Discrete Conformal Metric Deformation

## Definition (Conformal Deformation)

Given discrete conformal factor function $u : V(\mathcal{T}) \to \mathbb{R}$, hyperbolic vertex scaling is defined as $y := u * l$,

$$\sinh\frac{y_k}{2} = e^{\frac{u_i}{2}}\sinh\frac{l_k}{2}e^{\frac{u_j}{2}}$$

## Lemma (Symmetry)

*The symmetric relations holds:*

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i} = \frac{C_i + C_j - C_k - 1}{A(C_k + 1)}$$

*where* $S_k = \sinh y_k, C_k = \cosh y_k$.

# Discrete Hyperbolic Entropy Energy

## Definition (Hyperbolic Entropy Energy)

$$E_f(u_i, u_j, u_k) = \int^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

The Hessian matrix of the entropy energy is:

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{S_1}{C_1+1} & \frac{S_1}{C_1+1} \\ \frac{S_2}{C_2+1} & 0 & \frac{S_2}{C_2+1} \\ \frac{S_3}{C_3+1} & \frac{S_3}{C_3+1} & 0 \end{pmatrix}$$

which is strictly convex.

# Discrete Entropy Energy on a Mesh

## Definition (Entropy Energy)

The entropy energy on a triangle mesh with hyperbolic background geometry equals to

$$E(\mathbf{u}) = \int^{\mathbf{u}} \sum_i (\bar{K}_i - K_i) du_i$$

## Definition (Hyperbolic Ricci Flow)

Hence the discrete hyperbolic surface Ricci flow is defined as:

$$\frac{du_i(t)}{dt} = \bar{K}_i - K_i(t),$$

# Uniformizaton of High Genus Surface

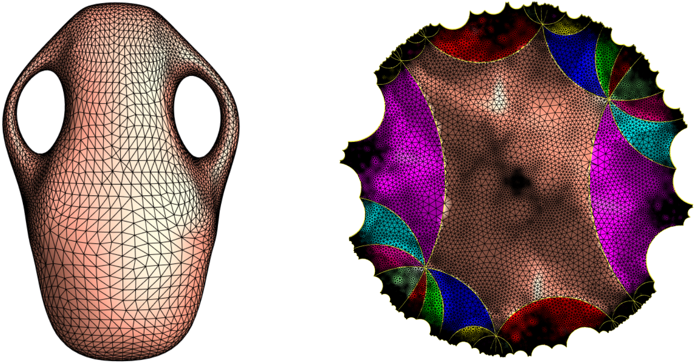

Figure: Uniformization of a genus two surface.

# Uniformization

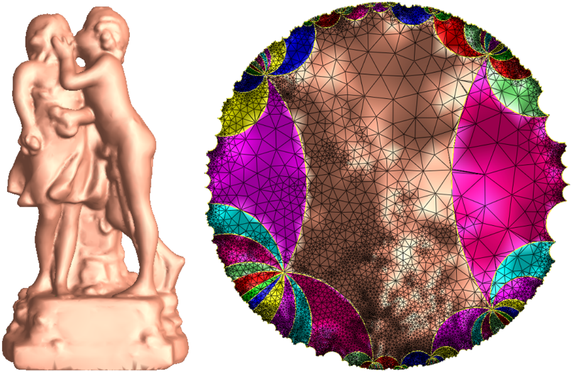

Figure: Uniformization of a genus three surface.

# Uniformization

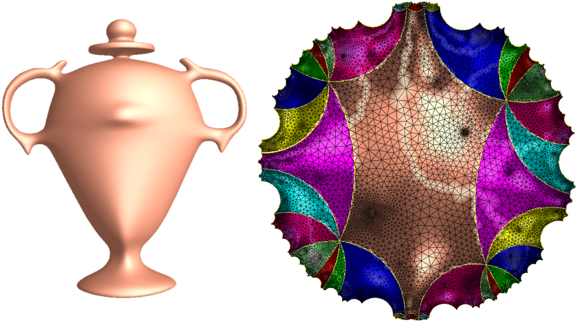

Figure: Uniformization of a genus two surface.

# Final Project

You can also choose other topics, which are related to computational conformal geometry and can demonstrate your talence and skills. The project is due within one month. The solution is required to be written in generic C++ with a detailed technical report to describe your design of data structures, algorithms, potential applications and improvement direction.