Phase Shifting Structured Light Method

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Stereo-vision with Structured Light

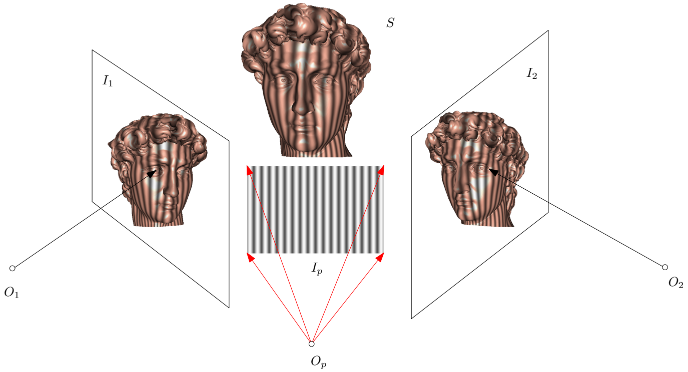
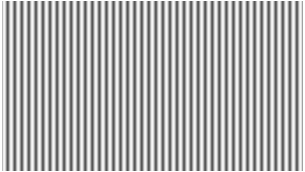
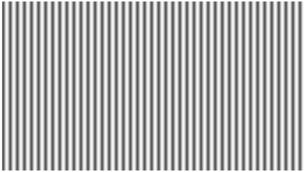


Figure: The fringe pattern for the digital projector or the LCD display. The left and right camera optical centers and image planes are (O_1, I_1) and (O_2, I_2) respectively. The projector optical center and image plane are (O_p, I_p).

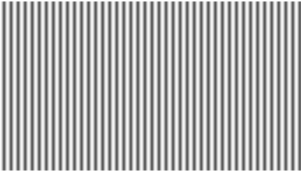
Projector Fringe Pattern



(a). $H_1(x, y)$



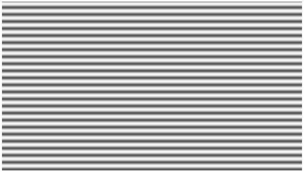
(b). $H_2(x, y)$



(c). $H_3(x, y)$

(d). $V_1(x, y)$

(e). $V_2(x, y)$



(f). $V_3(x, y)$

Figure: The fringe pattern for the digital projector or the LCD display.

Projector Fringe Pattern

The horizontal fringe patterns for the projector are given by

$$H_k(u,v) = a + b\cos\left(\Phi(u,v) + \Delta\varphi_k\right), \quad k = 1,2,3 \tag{1}$$

where $a = 128$ and $b = 127$,

$$(\Delta\varphi_1, \Delta\varphi_2, \Delta\varphi_3) = \left(-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right), \tag{2}$$

and the absolute phase

$$\Phi(u,v) = \frac{2\pi u}{\lambda}, \tag{3}$$

where λ is the wavelength.

Projector Fringe Pattern

The vertical fringe patterns for the projector are given by

$$V_k(u,v) = a + b\cos\left(\Phi(u,v) + \Delta\varphi_k\right), \quad k = 1,2,3 \tag{4}$$

where $a = 128$ and $b = 128$,

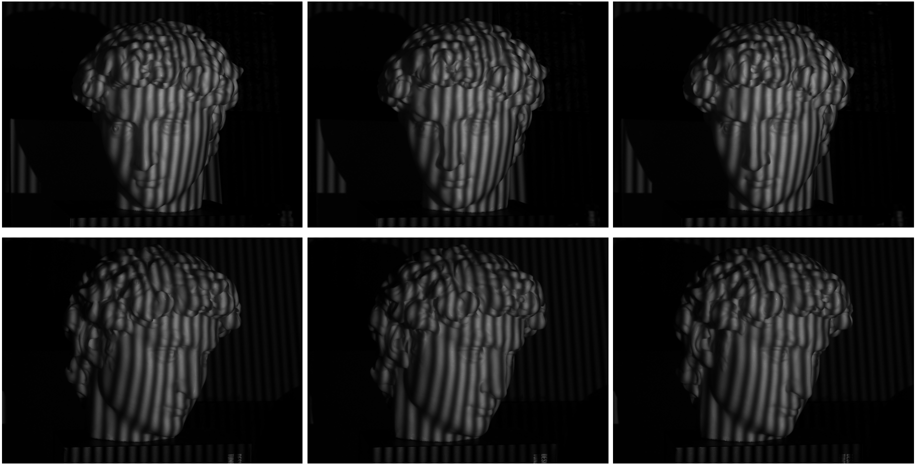
$$(\Delta\varphi_1, \Delta\varphi_2, \Delta\varphi_3) = \left(-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right), \tag{5}$$

and the absolute phase

$$\Phi(u,v) = \frac{2\pi v}{\lambda}, \tag{6}$$

where λ is the wavelength.

Fringe Images



(a). $I_1(x, y)$ (b). $I_2(x, y)$ (c). $I_3(x, y)$

Figure: Fringe images: the top row shows the images captured by the left camera, the bottom row show shows those by the right camera.

Image Decomposition

The image intensity is formulated as

$$
\begin{aligned}
I_1(x,y) &= I'(x,y) + I''(x,y)\cos(\Phi(x,y) - 2\pi/3)\\
I_2(x,y) &= I'(x,y) + I''(x,y)\cos(\Phi(x,y))\\
I_3(x,y) &= I'(x,y) + I''(x,y)\cos(\Phi(x,y) + 2\pi/3)
\end{aligned} \tag{7}
$$

where $I'(x,y)$ is the ambient, $I''(x,y)$ the intensity modulation, $\Phi(x,y)$ the absolute phase.

Image Decomposition

The ambient, the modulation and the relative phase can be obtained by

$$
\begin{aligned}
I'(x,y) &= \frac{1}{3}[I_1(x,y) + I_2(x,y) + I_3(x,y)] \\
I''(x,y) &= \frac{1}{3}\sqrt{3(I_1 - I_3)^2 + (2I_2 - I_1 - I_3)^2} \\
\varphi(x,y) &= \tan^{-1}\frac{\sqrt{3}(I_1 - I_3)}{2I_2 - I_1 - I_3}
\end{aligned} \tag{8}
$$

where the relative phase $\varphi(x,y)$ is from $-\pi$ to π,

$$\varphi(x,y) = \Phi(x,y) \mod 2\pi.$$

Image Decomposition

ambient $I'(x, y)$ modulation $I''(x, y)$ wrapped phase $\varphi(x, y)$

Figure: The ambient, modulation and wrapped phase computed from the fringe images in Fig. 3.

Phase Unwrapping

Problem (Phase Unwrapping)

Given an image $\mathcal{I}$ of m rows and n columns, each pixel position is represented as a pair of indices (i,j), where $1 \leq i \leq m$, $1 \leq j \leq n$. We use $p \in [1,m] \times [1,n]$ to represent a point in the image plane. The wrapped phase at the pixel p is denoted as φ_p, the wrap uncount at p as k_p, the unwrapped phase Φ_p, then

$$\Phi_p = \varphi_p + 2\pi k_p, \quad \forall p \in \mathcal{I}. \tag{9}$$

The wrap count function $k : \mathcal{I} \to \mathbb{Z}$ is the unkown function.

Phase Unwrapping - Multiple Wavelength Method

One can use multiple projection fringe pattern with different wavelengths to recover the absolute phase. The wavelength λ_i's satisfy the relation

$$\lambda_i = 2\lambda_{i+1}, \quad i = 0, 1, 2, \ldots, n.$$

λ_0 is big enough to cover the whole scanning range, so the relative phase φ_0 equals the absolute phase Φ_0, $\Phi_0 \leftarrow \varphi_0$. By the relation

$$\Phi_i(p)\lambda_i = \Phi_{i+1}(p)\lambda_{i+1} \implies 2\Phi_i = \Phi_{i+1}$$

We obtain

$$\Phi_{i+1} - \varphi_{i+1} = 2\Phi_i - \varphi_{i+1} \implies k_{i+1} = \text{round}\left(\frac{\Phi_i}{\pi} - \frac{\varphi_{i+1}}{2\pi}\right)$$

and

$$\Phi_{i+1} = 2\pi k_{i+1} + \varphi_{i+1}$$

Multiple Wavelength

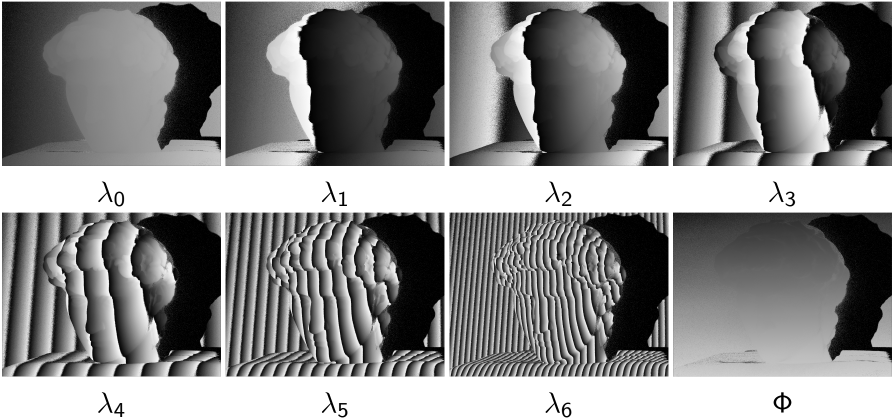


Figure: Multiple wavelength method.

Phase Unwrapping - Double Wavelength Method

One can use double projection fringe pattern with different wavelengths to recover the absolute phase. The wavelengths are λ_1 and λ_2, $\lambda_1 < \lambda_2$, then

$$\Phi_1(p) - \Phi_2(p) = \frac{2\pi u}{\lambda_1} - \frac{2\pi u}{\lambda_2} = \frac{2\pi u}{\lambda_{eq}} = \Phi_{eq} \tag{10}$$

where λ_{eq} is the equivalent wavelength,

$$\lambda_{eq} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \tag{11}$$

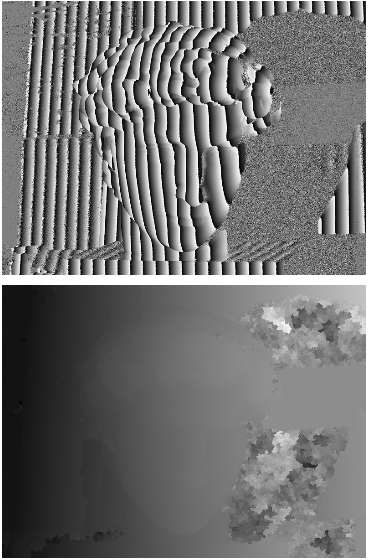
Once can choose λ_1 and λ_2 to be close enough, such that λ_{eq} covers the whole scanning range. So

$$\Phi_{eq} = \varphi_{eq} = \varphi_1 - \varphi_2 \pmod{2\pi},$$

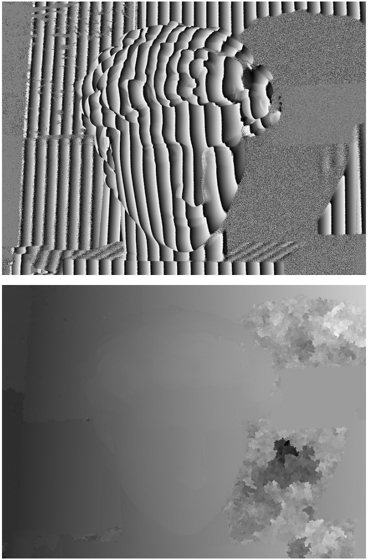
and

$$\Phi_{eq}\lambda_{eq} = \Phi_i \lambda_i \implies k_i = \frac{1}{2\pi}\left(\Phi_{eq}\frac{\lambda_{eq}}{\lambda_i} - \varphi_i\right)$$

Double Wavelength Method



φ and Φ $\lambda_1 = 18$



φ and Φ $\lambda = 18.5$

Phase Unwrap - Quality Guided Path Follower

Treat the modulation as the quality of the pixel.

1. Pick the pixel p with the highest quality as the base pixel, $\Phi(p) \leftarrow \varphi(p)$; insert p to the priority queue Q;
2. while the Q is not empty
   1. $p \leftarrow Q.pop()$;
   2. Find each unprocessed neighbor pixel q of p, unwrap the phase of q
   $$k(q) \leftarrow \text{argmin}_{k \in Z} (\Phi(p) - (\varphi(q) + 2\pi k))^2;$$
   $$\Phi(q) \leftarrow \varphi(q) + 2\pi k(q);$$
   3. Insert q to the priority queue Q

The quality guided path follower is vulnerable to bad pixels.

Phase Unwrap - Path Follower

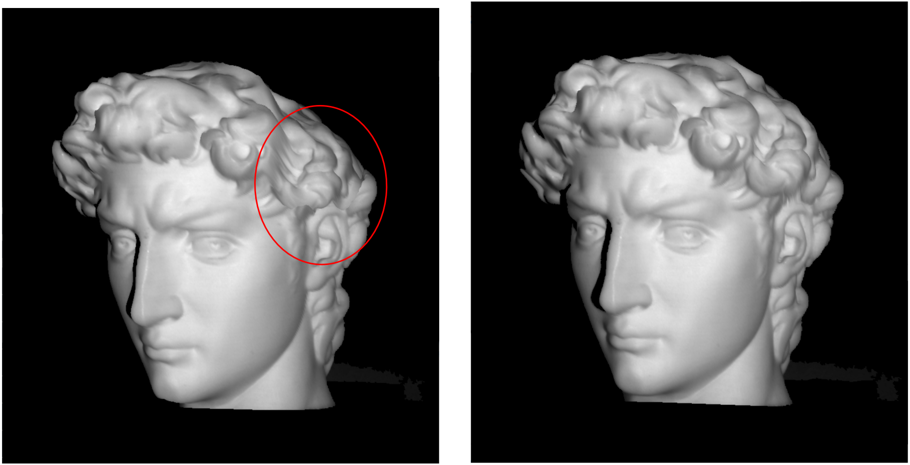


Figure: phase unwrapping mistake.

Phase Unwrap - Path Follower

Figure: phase unwrapping mistake.

Phase Unwrapping - Energy Method

The energy for wrap count function $k : \mathcal{I} \to \mathbb{Z}$ is defined as

$$E(k) := \sum_{e_{pq}} |\Phi_p - \Phi_q|^2, \tag{12}$$

where e_{pq} is the edge connecting the pixel p and the pixel q.

$$\begin{aligned} E(k) &:= \sum_{e_{pq}} E_{pq} \\ E_{pq} &:= |(\varphi_p + 2\pi k_p) - (\varphi_q + 2\pi k_q)|^2 \\ &= \sum_p k_p \sum_{e_{pq}} 4\pi(\varphi_p - \varphi_q) + \sum_{e_{pq}} 4\pi^2 (k_p - k_q)^2. \end{aligned}$$

Phase Unwrapping - Energy Method

The energy based phase unwrapping problem is formulated as a combinatorial optimization method using Ising model, which is a mathematical model of ferromagnetism in statistical mechanics to study phase transition.

Figure: Ising model for phase transition.

Phase Unwrapping - Energy Method

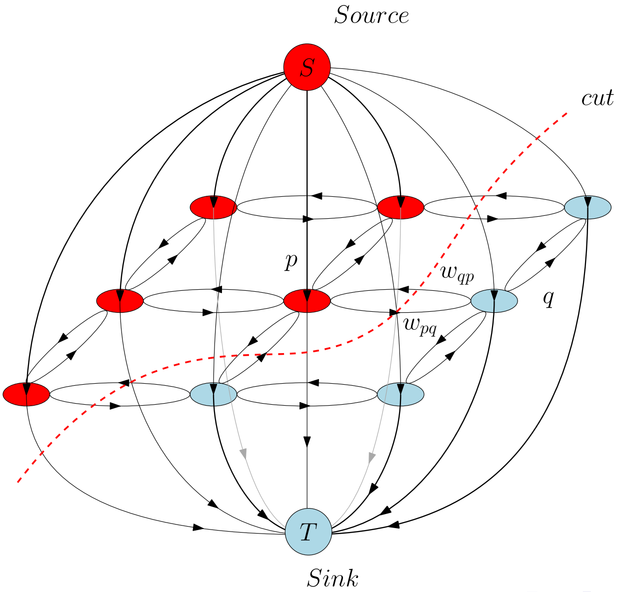
The phase unwrapping energy statisfies the following property:

$$E_{pq}(1,0) + E_{pq}(0,1) - E_{pq}(0,0) - E_{pq}(1,1) \geq 0, \tag{13}$$

the energy is *sub-modular*. So the energy is graph representable.

- Each pixel p is represented by a node n_p;
- Each pair of adjacent pixels p, q are represented as two oriented edges e_{pq} and e_{qp};
- One source noce is added as S; One sink is added as T;
- For each pixel p, an oriented edge e_{Sp} is added, and an oriented edge e_{pT} is added.

Phase Unwrapping - Energy Method



Phase Unwrapping - Energy Method

Definition (Network)

A network consists of

- a finite directed graph $N = (V, E)$, where V denotes the finite set of vertices and $E \subset V \times V$ is the set of directed edges;
- a source $s \in V$ and a sink $t \in V$;
- a capacity function, which is a mapping $c : E \to \mathbb{R}^+$ denoted by $c(u, v)$ for $(u, v) \in E$, representing the maximum amount of flow that can pass through an edge.

Phase Unwrapping - Energy Method

Definition (Flows)

A flow through a network is a mapping $f : E \to \mathbb{R}^+$ denoted as $f(u, v)$, subject to the following constraints:

1. Capacity constraint: for every edge $(u, v) \in E$, $f(u, v) \leq c(u, v)$;
2. Conservation of flows: for each vertex v apart from s and t, the following equality holds:

$$\sum_{(u,v)\in E} f(u, v) = \sum_{(v,w)\in E} f(v, w).$$

The value of flow is defined by

$$|f| = \sum_{(s,v)\in E} f(S, v) = \sum_{(v,t)\in E} f(v, T).$$

Definition (Cut)

An $s-t$ *cut* $C=(S,T)$ is a partition of V, such that $s\in S$ and $t\in T$. The *cut-set* X_C of a cut C is the set of oriented edges that connect the source part to the sink part:

$$X_C := \{(u,v)\in E : u\in S, v\in T\} = (S\times T)\cap E.$$

The *capacity* of an $s-t$ cut is the sum of the capacities of the edges in its cut-set,

$$c(S,T) = \sum_{(u,v)\in X_C} c(u,v).$$

Phase Unwrapping - Energy Method

Problem (Max Flow)

Given a network, maximize $|f|$, *route as much flow as possible from* s *to* t.

Problem (Min Cut)

Given a network, minimize $c(S, T)$, *determine* S *and* T *such that the capacity of the s-t cut is minimal.*

Theorem (Max-flow min-cut)

The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

Ford-Fulkerson Algorithm

Define the residue network $G_f(V, E_f)$ to be the network with capacity $c_f(u,v) = c(u,v) - f(u,v)$ and no flow.

Ford-Fulkerson Algorithm

Input: Given a network $G = (V, E)$ with flow capacity c, a source node s and a sink node (t);

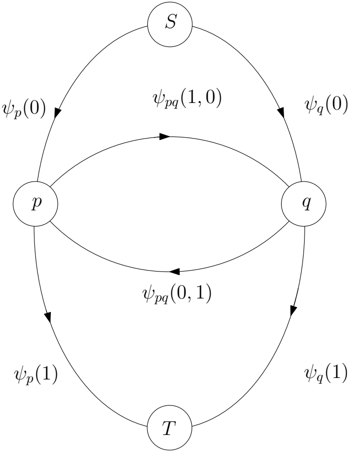
Output: Compute a flow f from s to t of maximumm value;

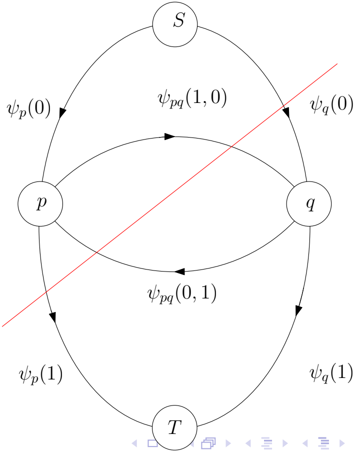
1. $f(u,v) \leftarrow 0$ for all edges (u,v)
2. While there is a path p from s to t in G_f, such that $c_f(u,v) > 0$ for all edges $(u,v) \in p$; (p can be found using BFS or DFS)
    1. Find $c_f(p) = \min\{c_f(u,v) : (u,v) \in p\}$
    2. For each edge $(u,v) \in p$

$$f(u,v) \leftarrow f(u,v) + c_f(p)$$
$$f(v,u) \leftarrow f(v,u) - c_f(p)$$

Edge energy

$$E_{pq} = k_p 4\pi(\varphi_p - \varphi_q) + k_q 4\pi(\varphi_q - \varphi_p) + 4\pi^2(k_p - k_q)^2 + (\varphi_p - \varphi_q)^2$$
$$= \psi_p(k_p) + \psi_q(k_q) + \psi_{pq}(k_p, k_q).$$





Combinatorial Optimization

Graph Construction

1. For each edge $(p,q) \in \mathcal{I}$, construct a graph $G_{p,q}$ with a source, since, the edge capacities are given by $\psi_p(k_p)$, $\psi_q(k_q)$ and $\psi_{pq}(k_p,k_q)$;
2. The graph for the whole image $G_{\mathcal{I}}$ is obtained by merging the edge graphs

$$G_{\mathcal{I}} = \bigcup_{(p,q)\in E} G_{pq},$$

   the capacities on the same edge are accumulated together;

Global Optimization

1. At each step, the edge capacity is updated by the one-step optimal wrap count;
2. The optimal wrap count (for one step) is obtained by the max-flow/min-cut algorithm;
3. The global optimal wrap count can be found by iterations.

Phase Unwrapping- Energy Method

Figure: Computational example.

Phase Unwrapping- Energy Method

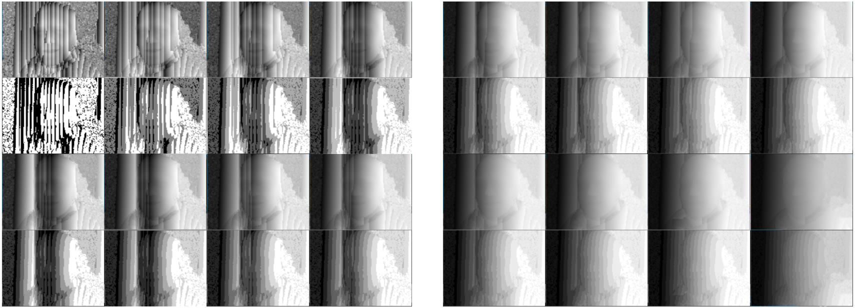


Figure: Phase unwrapping based on combinatorial optimization.

Phase Unwrapping- Energy Method

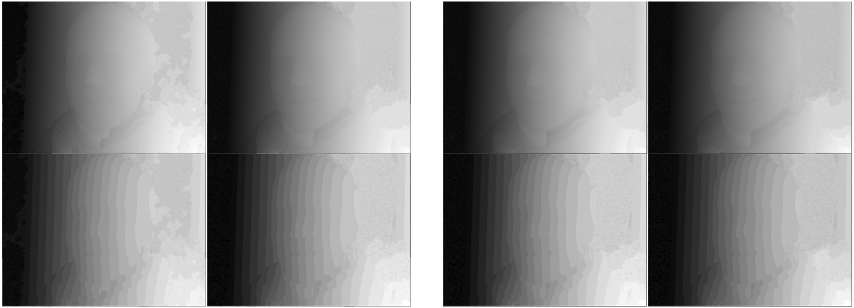


Figure: Phase unwrapping based on combinatorial optimization.

Phase Unwrapping- Energy Method

Figure: Reconstructed point cloud.

Phase Unwrapping- Energy Method

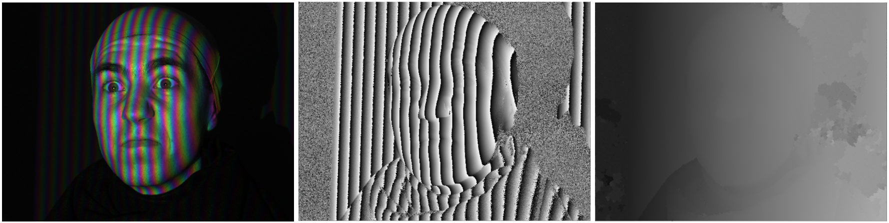


Figure: phase shifting structured light.

Phase Unwrapping- Energy Method

Figure: Reconstructed point cloud.

Energy Method - Poisson Equation

One can assume the scanned surface is smooth, the smoothness is measured by the harmonic energy. Since $\Phi(p) = \varphi + 2\pi k_p$, locally, we obtain $\nabla\Phi = \nabla\varphi$. Due to the noise, $\nabla\varphi$ may not be a gradient field. By Hodge decomposition,

$$\nabla\varphi = d\Phi + \nabla \cdot v, \tag{14}$$

for some vector field v, namely $\nabla \cdot v$ is the divergence free component. Therefore, we obtain

$$\Delta\Phi(p) = \nabla \cdot \nabla\varphi(p). \tag{15}$$

The absolute phase Φ can be obtained by solving the Poisson equation.

Energy Method - Poisson Equation

We sample Ω by regular grid points with horizontal and vertical step lengths h_x and h_y respectively, the finite difference operators are defined as

$$\mathcal{D}^2_{xx} u_{ij} = \frac{1}{h_x^2}(u_{i+1,j} + u_{i-1,j} - 2u_{i,j})$$
$$\mathcal{D}^2_{yy} u_{ij} = \frac{1}{h_y^2}(u_{i,j+1} + u_{i,j-1} - 2u_{i,j})$$
$$\mathcal{D}^2_{xy} u_{ij} = \frac{1}{4h_x h_y}(u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1})$$

Then we obtain discrete Poisson equation:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = \rho_{i,j},$$

with Neumann boundary condition.

Energy Method - Poisson Equation

The 2D Discrete Cosine Transformation is defined as

$$\tilde{u}(m,n) = c(m,n)\sum_{i,j}\sum_{i,j}\cos\frac{(2i+1)m\pi}{2M}\cos\frac{(2j+1)n\pi}{2N},$$

the Inverse Discrete Cosine Transformation is given by

$$u(i,j) = \sum_{m,n} c(m,n)\tilde{u}(m,n)\cos\frac{(2i+1)m\pi}{2M}\cos\frac{(2j+1)n\pi}{2N},$$

where $m, i = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots, N-1$,

$$c(m,n) = \begin{cases} \frac{\sqrt{2}}{\sqrt{MN}} & m = 0, n = 0 \\ \frac{2}{\sqrt{MN}} & \text{otherwise} \end{cases}$$

Energy Method - Poisson Equation

Given a discrete Poisson equation with Neumann boundary condition:

$$\begin{cases} \Delta u &= \rho \\ \frac{\partial u}{\partial n} &= 0 \end{cases}$$

Let $\tilde{\rho} = DCT(\rho)$, $\tilde{u} = DCT(u)$, then we have

$$\tilde{u}(m,n) = \frac{\tilde{\rho}(m,n)}{2[\cos\frac{m\pi}{M} + \cos\frac{n\pi}{N} - 2]}$$

Different solutions differ by a constant. Let $\tilde{u}(0,0) = 0$, we get the unique solution. The DCT, IDCT can be computed using FFT on GPUs.

Phase Unwrapping- Energy Method

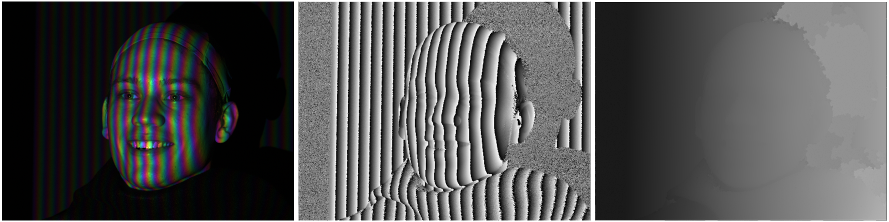
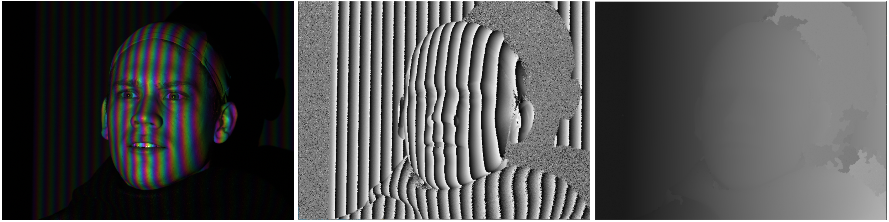


Figure: phase shifting structured light.

Phase Unwrapping- Energy Method

Figure: Reconstructed point cloud.

Phase Unwrapping- Energy Method



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Phase Unwrapping- Energy Method

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