

# General Discrete Surface Curvature Flows

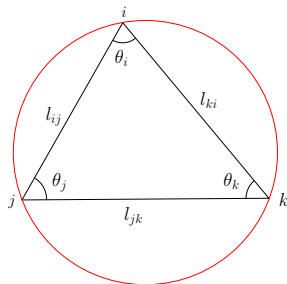
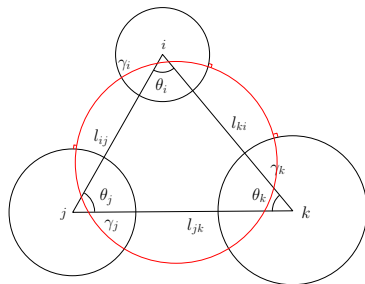
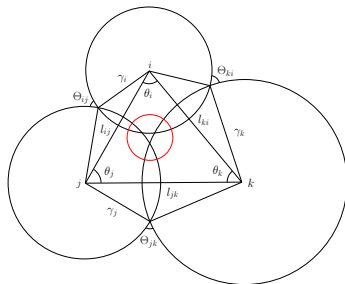
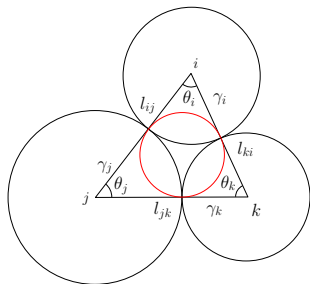
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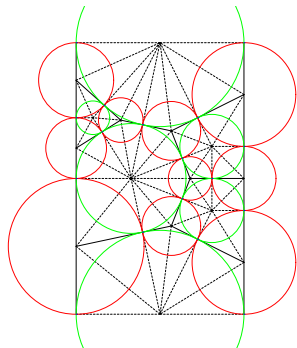
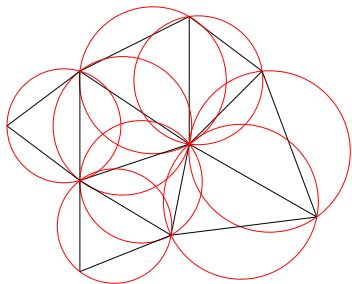
August 21, 2022

# General Derivative Cosine Laws

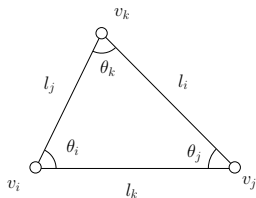
# Different Schemes



# Different Schemes



# Derivative Cosine law



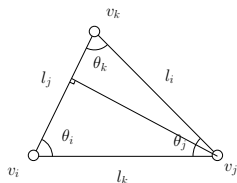
$$A = l_j l_k \sin \theta_i$$

$$\frac{\partial}{\partial l_i} (2l_j l_k \cos \theta_i) = \frac{\partial}{\partial l_i} (l_j^2 + l_k^2 - l_i^2)$$

$$-2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_i} = -2l_i$$

$$\frac{d\theta_i}{dl_i} = \frac{l_i}{A}$$

# Derivative Cosine law



$$l_j = l_i \cos \theta_k + l_k \cos \theta_i$$

$$\frac{\partial}{\partial l_j} (2l_j l_k \cos \theta_i) = \frac{\partial}{\partial l_j} (l_j^2 + l_k^2 - l_i^2)$$

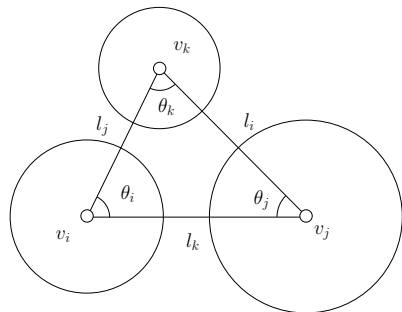
$$2l_j = 2l_k \cos \theta_i - 2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_j}$$

$$\frac{d\theta_i}{dl_j} = \frac{l_k \cos \theta_i - l_j}{A}$$

$$= -\frac{l_i \cos \theta_k}{A}$$

$$= -\frac{d\theta_i}{dl_j} \cos \theta_k$$

# Derivative Cosine law



$$l_{ij}^2 = r_i^2 + r_j^2 + 2r_i r_j \cos \theta_k$$

$l_{ij}$  inversive distance.

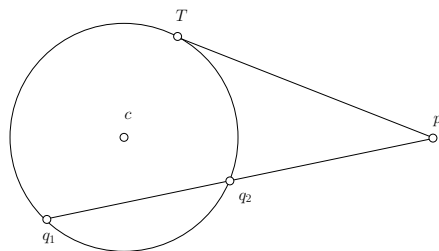
$$\begin{aligned} \frac{\partial}{\partial r_j} l_{ij}^2 &= \frac{\partial}{\partial r_j} (r_j^2 + r_k^2 + 2r_j r_k \cos \theta_k) \\ 2l_{ij} \frac{dl_{ij}}{dr_j} &= 2r_j + 2r_k \cos \theta_k \\ \frac{dl_{ij}}{dr_j} &= \frac{2r_j^2 + 2r_j r_k \cos \theta_k}{2l_{ij} r_j} \\ &= \frac{r_j^2 + r_k^2 + 2r_j r_k \cos \theta_k + r_j^2 - r_k^2}{2l_{ij} r_j} \\ &= \frac{l_{ij}^2 + r_j^2 - r_k^2}{2l_{ij} r_j} \end{aligned}$$

# Cosine law

Let  $u_i = \log r_i$ , then  $\frac{d\theta}{du} = \frac{d\theta}{dl} \frac{dl}{dr} \frac{dr}{du}$

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \theta_3 & \cos \theta_2 \\ \cos \theta_3 & -1 & \cos \theta_1 \\ \cos \theta_2 & \cos \theta_1 & -1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{l_1^2 + r_2^2 - r_3^2}{2l_1 r_2} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1 r_3} \\ \frac{l_2^2 + r_1^2 - r_3^2}{2l_2 r_1} & 0 & \frac{l_2^2 + r_3^2 - r_1^2}{2l_2 r_3} \\ \frac{l_3^2 + r_1^2 - r_2^2}{2l_3 r_1} & \frac{l_3^2 + r_2^2 - r_1^2}{2l_3 r_2} & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

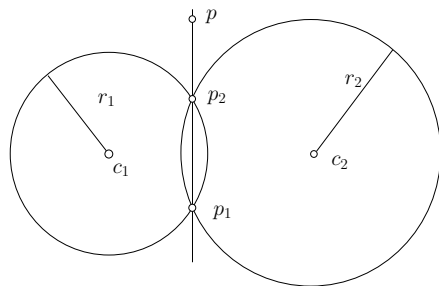




## Power

Suppose a point  $p$  is not coincident of the center of a circle  $\mathbf{c} = (c, r)$  on the plane, the line through  $p$  intersects the circle at  $q_1$  and  $q_2$ ,  $T$  is the tangent point, then the power of  $p$  with respect to the circle is

$$\begin{aligned} \text{pow}(p, \mathbf{c}) &= |pq_1||pq_2| \\ &= |pT|^2 \\ &= |pc|^2 - r^2. \end{aligned}$$



$$\text{pow}(p, \mathbf{c}_1) = |pp_1||pp_2| = \text{pow}(p, \mathbf{c}_2)$$

## Equi-Power line

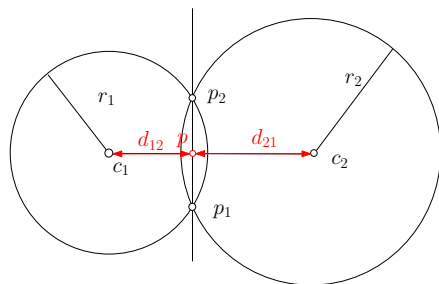
Suppose there are two circles  $\mathbf{c}_1 = (c_1, r_1)$ ,  $\mathbf{c}_2 = (c_2, r_2)$ , the equi-power line is the locus

$$\text{pow}(p, \mathbf{c}_1) = \text{pow}(p, \mathbf{c}_2).$$

The equation of  $p$  is

$$|p - c_1|^2 - r_1^2 = |p - c_2|^2 - r_2^2.$$

If two circles intersect at  $p_1$  and  $p_2$ , then the line through them is the equi-power line.



$$\text{pow}(p, \mathbf{c}_1) = \text{pow}(p, \mathbf{c}_2)$$

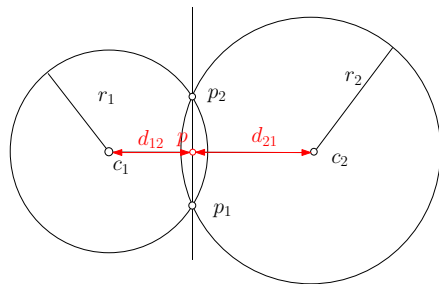
$$d_{12}^2 - r_1^2 = d_{21}^2 - r_2^2$$

Suppose there are two circles  $\mathbf{c}_k = (c_k, r_k)$ , the line through  $c_1$  and  $c_2$  intersects the equi-power line at the point  $p$ . Assume the length between  $c_1$  and  $c_2$  is  $l$ . The distance from  $p$  to  $c_2$  is denoted as  $d_{21}$ , then

$$d_{12} = \frac{l^2 + r_1^2 - r_2^2}{2l}$$

$$d_{21} = \frac{l^2 + r_2^2 - r_1^2}{2l}$$

obviously,  $d_{12} + d_{21} = l$ .



$$\text{pow}(p, \mathbf{c}_1) = -|pp_1||pp_2|$$

$$\text{pow}(p, \mathbf{c}_2) = -|pp_1||pp_2|$$

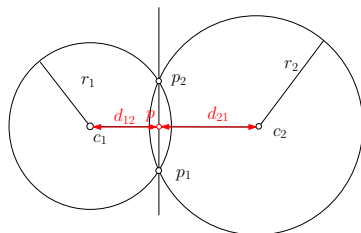
$$\text{pow}(p, \mathbf{c}_1) = \text{pow}(p, \mathbf{c}_2)$$

compute the power of  $p$  with respect to two circles

$$\text{pow}(p, \mathbf{c}_1) = d_{12}^2 - r_1^2$$

$$\text{pow}(p, \mathbf{c}_2) = d_{21}^2 - r_2^2$$

$$\begin{aligned} d_{12}^2 - d_{21}^2 &= (d_{12} + d_{21})(d_{12} - d_{21}) \\ &= \frac{r_1^2 - r_2^2}{1} = r_1^2 - r_2^2. \end{aligned}$$



## Lemma

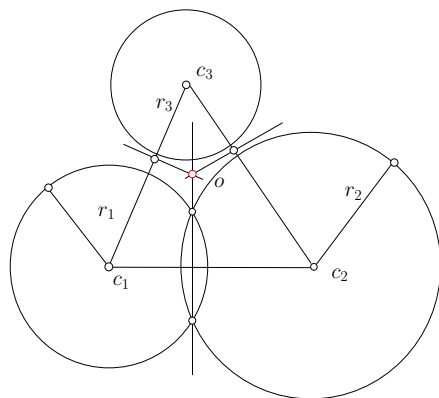
*The equi-power line is orthogonal to the line connecting the centers.*

## Proof.

Define a function  $\phi(p) = \text{pow}(p, c_1) - \text{pow}(p, c_2)$ ,

$$\begin{aligned}\phi(p) &= \langle p - c_1, p - c_1 \rangle - r_1^2 - \langle p - c_2, p - c_2 \rangle + r_2^2 \\ d\phi(p) &= \langle dp, c_2 - c_1 \rangle\end{aligned}$$

so  $\nabla\phi = c_2 - c_1$ , orthogonal to the level sets of  $\phi$ . The equi-power line is the 0-level set of  $\phi$ .



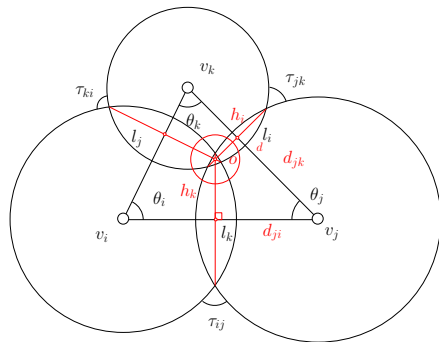
Given three circles  $\mathbf{c}_k$ ,  $k = 1, 2, 3$ , then three equi-power lines intersect at one point  $o$ , which is called the *power center*,

The equi-power lines of  $\mathbf{c}_1, \mathbf{c}_2$  and  $\mathbf{c}_1, \mathbf{c}_3$  intersects at the point  $o$ . Then

$$\text{pow}(o, \mathbf{c}_1) = \text{pow}(o, \mathbf{c}_2) = \text{pow}(o, \mathbf{c}_3)$$

so  $o$  is also on the equi-power line of  $\mathbf{c}_2, \mathbf{c}_3$ .

# Power Center



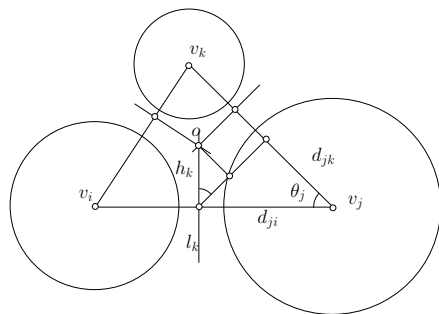
There are 3 circles  $\mathbf{c}_k = (c_k, r_k)$ , the power center  $o$  is also the center of the unique circle  $(p, r)$ , which is orthogonal to all 3 circles.

$$\text{pow}(o, \mathbf{c}_k) = \langle o - c_k, o - c_k \rangle - r_k^2 = r^2,$$

so the power center is the center of the circle which is orthogonal to the 3 circles.

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$

# Derivative Cosine law



$$\frac{\partial}{\partial r_j} l_k^2 = \frac{\partial}{\partial r_j} (r_i^2 + r_j^2 + 2l_{ij}r_i r_j)$$

$$\text{pow}(o, \mathbf{c}_i) = \text{pow}(o, \mathbf{c}_j)$$

$$|ov_i|^2 - r_i^2 = |ov_j|^2 - r_j^2$$

$$|ov_i|^2 - |ov_j|^2 = r_i^2 - r_j^2$$

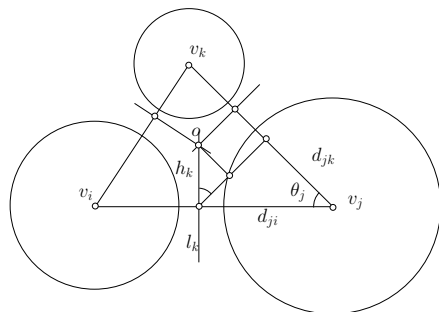
$$2l_k \frac{dl_k}{dr_j} = 2r_j + 2r_i l_{ij}$$

$$\begin{aligned} r_j \frac{dl_k}{dr_j} &= \frac{2r_j^2 + 2r_i r_j l_{ij}}{2l_k} \\ &= \frac{r_j^2 + 2r_i r_j l_{ij} + r_i^2 - r_i^2 + r_j^2}{2l_k} \\ &= \frac{l_k^2 + r_j^2 - r_i^2}{2l_k} \\ &= \frac{l_k^2 + |ov_j|^2 - |ov_i|^2}{2l_k} = d_{ji} \end{aligned}$$

Therefore in  $\Delta v_i v_j o$ ,  $\frac{dl_k}{du_j} = d_{ji}$ .



# Derivative Cosine law

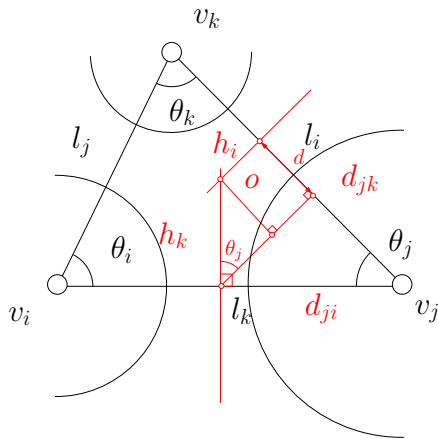


The distance from  $o$  to edge  $[v_i, v_j]$  is  $h_k$ .

## Theorem (Symmetry)

$$\begin{aligned}\frac{d\theta_i}{du_j} &= \frac{d\theta_j}{du_i} = \frac{h_k}{l_k} \\ \frac{d\theta_j}{du_k} &= \frac{d\theta_k}{du_j} = \frac{h_i}{l_j} \\ \frac{d\theta_k}{du_i} &= \frac{d\theta_i}{du_k} = \frac{h_j}{l_i}\end{aligned}$$

# Derivative Cosine law



$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

Proof.

$$\begin{aligned} \frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i} \frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k} \frac{\partial l_k}{\partial u_j} \\ &= \frac{\partial \theta_i}{\partial l_i} \left( \frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j} \cos \theta_j \right) \\ &= \frac{l_i}{A} (d_{jk} - d_{ji} \cos \theta_j) \\ &= \frac{dl_i}{l_i l_k \sin \theta_j} \\ &= \frac{h_k \sin \theta_j}{l_k \sin \theta_j} \\ &= \frac{h_k}{l_k} \end{aligned}$$

□

# Inversive Distance CP Metric - Local Rigidity

The Discrete Ricci energy of Inversive distance CP metric is convex, but the conformal factor space is non-convex. Therefore it has local rigidity, not global rigidity.



# Discrete Yamabe flow - Local Rigidity

The Discrete Ricci energy of discrete Yamabe flow is convex, but the conformal factor space is non-convex. Therefore it has local rigidity, not global rigidity.

# Extremal Length



Figure: The conformal module of a topological quadrilateral.

# Topological Annulus

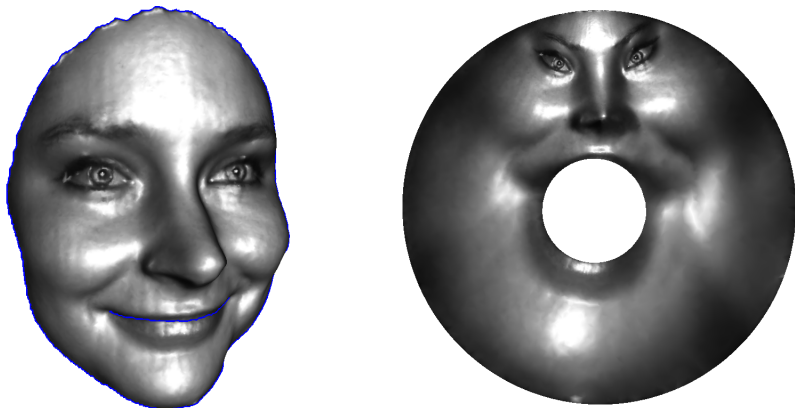


Figure: The conformal module of a topological annulus.

# Costa Minimal Surface

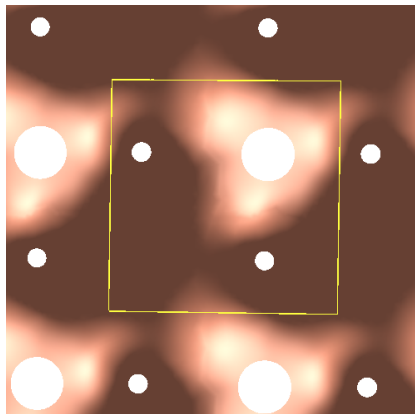
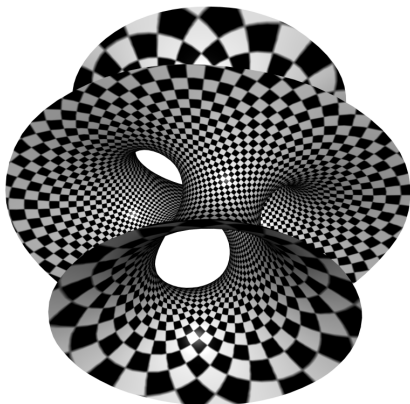


Figure: Costa minimal surface.



# Circle Packing and Square Packing

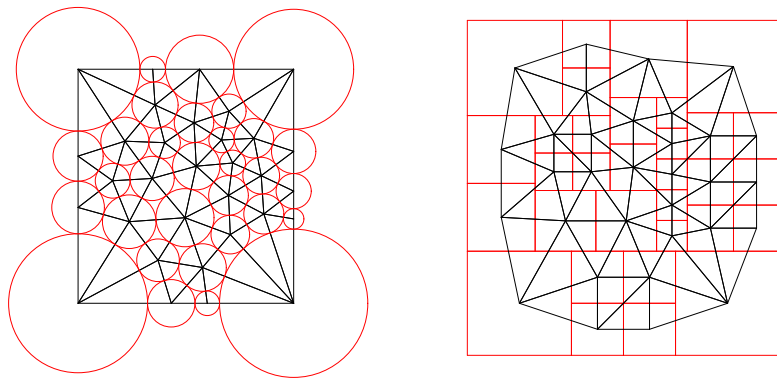


Figure: Circle packing and square packing.

# Circle Packing Art

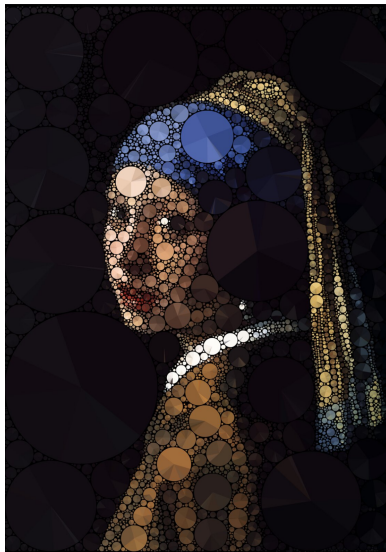


Figure: Girl with a Pearl Earring. (by Mario Klingemann) ◀ ☰ ▶ ☰ 🔍 ↻

# Circle Packing Art

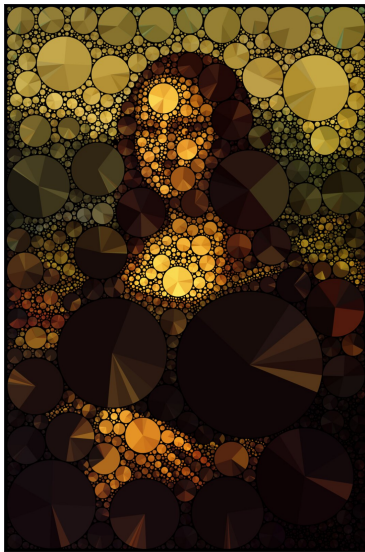


Figure: Mona Lisa. (by Mario Klingemann)

# Circle Packing Art



Figure: The Starry Night. (by Mario Klingemann)

# Hyperbolic Surface Ricci Flow

# Polyhedral Surface

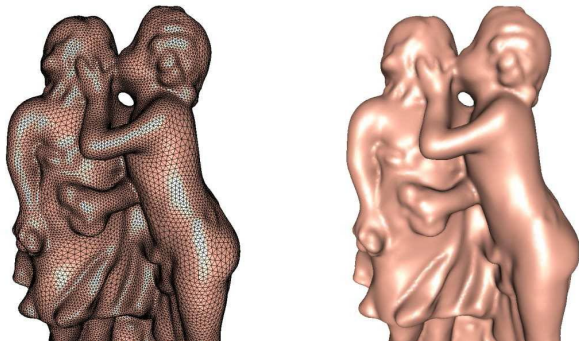


Figure: Polyhedral surface.

# Background Geometries

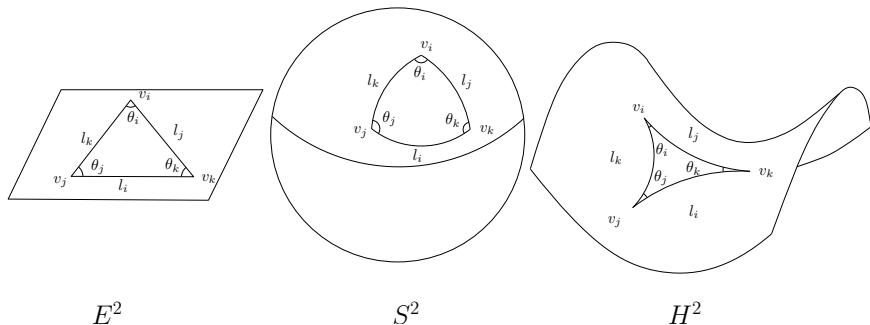


Figure: Constant curvature triangle.

We can glue hyperbolic or spherical triangles isometrically along the common edges to construct the triangle mesh. Then we say the surface is with hyperbolic or spherical background geometry.

# Hyperbolic Triangle

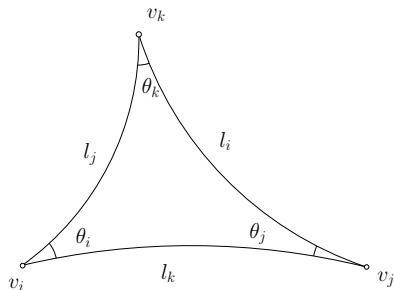


Figure: Hyperbolic triangle.

Cosine law:

$$\cos\theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

Sine law:

$$\frac{\sinh l_i}{\sin\theta_i} = \frac{\sinh l_j}{\sin\theta_j} = \frac{\sinh l_k}{\sin\theta_k}$$

Area

$$A = \frac{1}{2} \sinh l_j \sinh l_k \sin\theta_i$$



# Hyperbolic Derivative Cosine Law

## Lemma

*The hyperbolic derivative cosine law is represented as:*

$$\frac{\partial \theta_i}{\partial l_i} = \frac{\sinh l_i}{A}, \quad \frac{\partial \theta_i}{\partial l_j} = -\frac{\sinh l_i}{A} \cos \theta_k$$

Compared with Euclidean cosine law, we replace the edge lengths  $l_i$  by  $\sinh l_i$ .

## Definition (Discrete Curvature)

Given a discrete surface with hyperbolic background geometry  $(S, V, \mathcal{T}, l)$ , every triangle is a hyperbolic geodesic triangle, the vertex discrete curvature is defined as the angle deficit

$$K(v) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk}, & v \notin \partial S \\ \pi - \sum_{jk} \theta_i^{jk}, & v \in \partial S \end{cases}$$

## Theorem (Gauss-Bonnet)

*The discrete Gauss-Bonnet theorem is represented as:*

$$\sum_{v \notin \partial S} K(v) + \sum_{v \in \partial S} K(v) - \text{Area}(S) = 2\pi\chi(S)$$

# Discrete Conformal Metric Deformation

## Definition (Conformal Deformation)

Given discrete conformal factor function  $u : V(\mathcal{T}) \rightarrow \mathbb{R}$ , hyperbolic vertex scaling is defined as  $y := u * l$ ,

$$\sinh \frac{y_k}{2} = e^{\frac{u_i}{2}} \sinh \frac{l_k}{2} e^{\frac{u_j}{2}}$$

## Lemma (Symmetry)

*The symmetric relations holds:*

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{C_i + C_j - C_k - 1}{A(C_k + 1)}$$

where  $S_k = \sinh y_k$ ,  $C_k = \cosh y_k$ .

## Definition (Hyperbolic Entropy Energy)

$$E_f(u_i, u_j, u_k) = \int^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

The Hessian matrix of the entropy energy is:

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{S_1}{C_{1+1}} & \frac{S_1}{C_{1+1}} \\ \frac{S_2}{C_{2+1}} & 0 & \frac{S_2}{C_{2+1}} \\ \frac{S_3}{C_{3+1}} & \frac{S_3}{C_{3+1}} & 0 \end{pmatrix}$$

which is strictly convex.

# Discrete Entropy Energy on a Mesh

## Definition (Entropy Energy)

The entropy energy on a triangle mesh with hyperbolic background geometry equals to

$$E(\mathbf{u}) = \int^{\mathbf{u}} \sum_i (\bar{K}_i - K_i) du_i$$

## Definition (Hyperbolic Ricci Flow)

Hence the discrete hyperbolic surface Ricci flow is defined as:

$$\frac{du_i(t)}{dt} = \bar{K}_i - K_i(t),$$

which is the gradient flow of the discrete hyperbolic entropy energy. The strict concavity of the discrete entropy ensures the uniqueness of the solution to the flow. The existence is given by Gu-Luo-Sun using Teichmüller theory and hyperbolic geometry.

# Uniformization of High Genus Surface

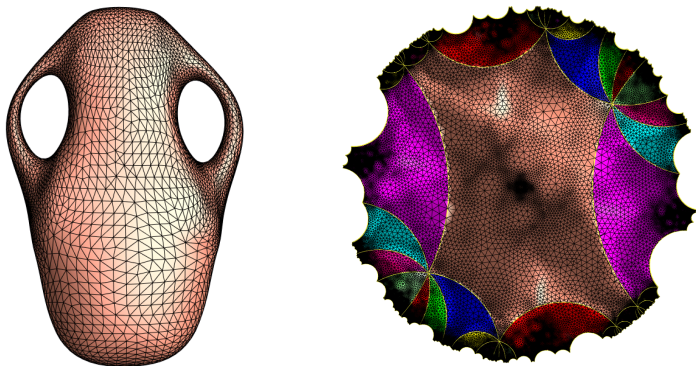


Figure: Uniformization of a genus two surface.

# Uniformization

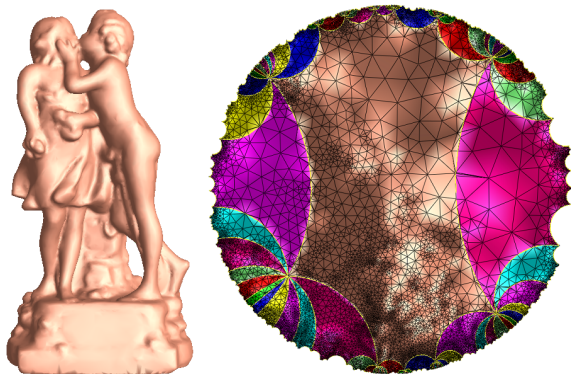


Figure: Uniformization of a genus three surface.

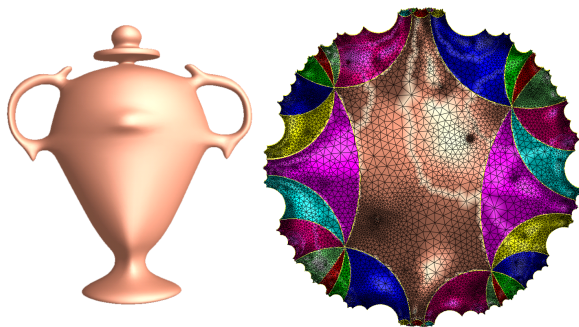


Figure: Uniformization of a genus two surface.



# Shortest Word

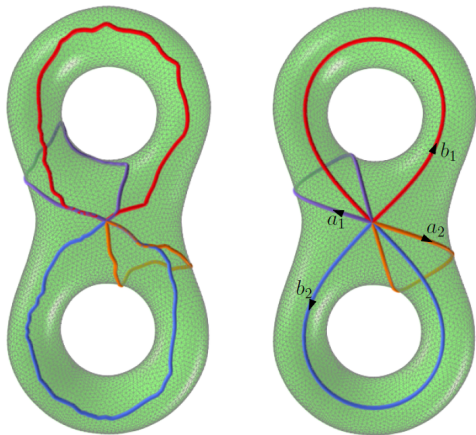
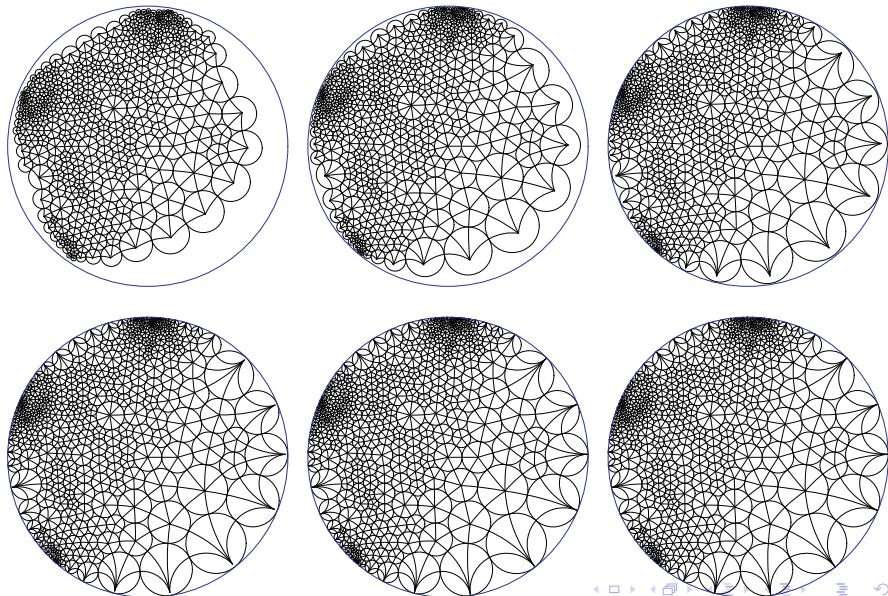


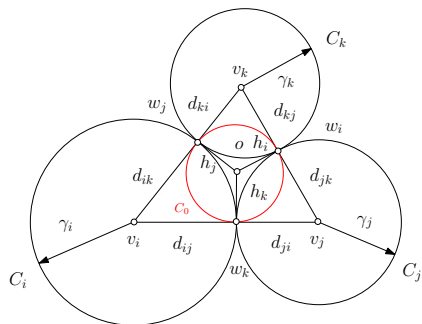
Figure: Shortest word problem.

# Discrete Riemann Mapping

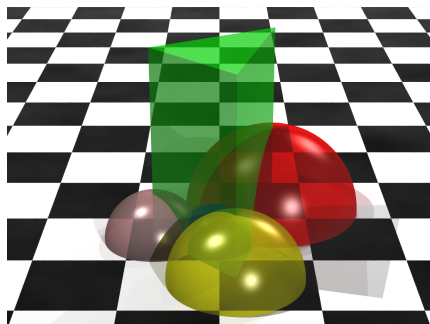


# Unified Discrete Surface Ricci Flow

# Unified Ricci Flow



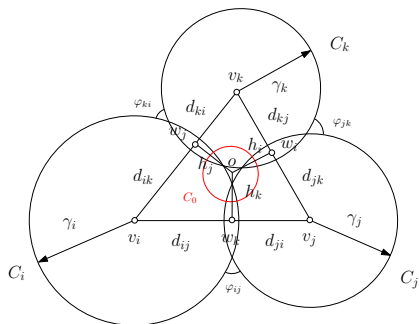
(a) Tangential CP



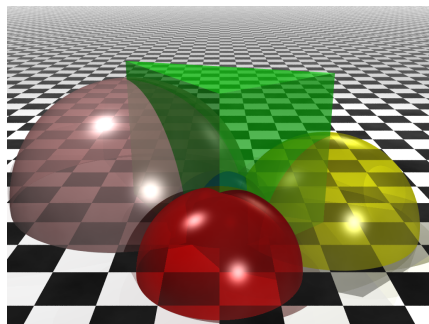
(b) Generalized Hyperbolic Tetrahedron,  $(\eta, \epsilon) = (1, 1)$

Figure: Tangential circle packing.

# Thurston's Circle Packing



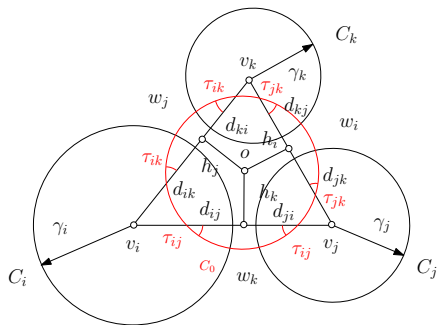
(a) Thurston's Circle packing



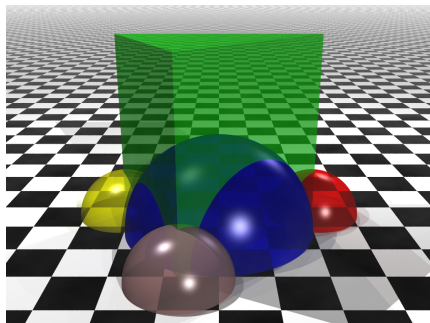
(b) Generalized Hyperbolic Tetrahedron,  $0 \leq \eta < 1, \epsilon = 1$

Figure: Thurston's circle packing.

# Inversive Distance Circle Packing

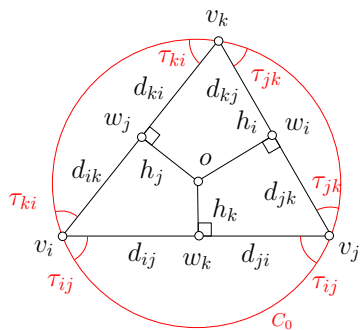


(c) Inversive distance CP

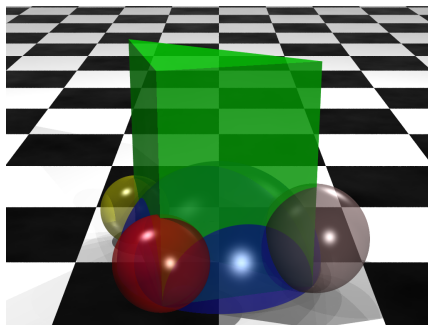


(d) Generalized Hyperbolic Tetrahedron,  $\eta > 1, \epsilon = 1$

Figure: Inversive distance circle packing.



(d) Yamabe flow

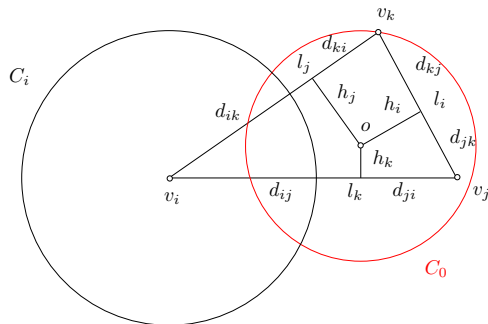


(e) Generalized Hyperbolic Tetrahedron,  $\eta > 0, \epsilon = 0$

Figure: Yamabe flow.







(f) mixed type

Figure: Mixed typed circle packing.

## Definition (Discrete Conformal Factor)

The discrete conformal factor is defined as  $u : V \rightarrow \mathbb{R}$ ,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{E}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

## Definition (Edge Length)

The edge lengths are given by

$$u_i = \begin{cases} l_{ij}^2 & = 2\eta_{ij}e^{u_i+u_j} + \varepsilon_i e^{2u_i} + \varepsilon_j e^{2u_j} & \mathbb{E}^2 \\ \cosh l_{ij} & = \frac{4\eta_{ij}e^{u_i+u_j} + (1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})}{(1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})} & \mathbb{H}^2 \\ \cos l_{ij} & = \frac{-4\eta_{ij}e^{u_i+u_j} + (1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})}{(1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})} & \mathbb{S}^2 \end{cases}$$

Scheme	$\varepsilon_i$	$\varepsilon_j$	$\eta_{ij}$
Tangential Circle Packing	+1	+1	+1
Thurston's Circle Packing	+1	+1	$[0, 1]$
Inversive Distance Circle Packing	+1	+1	$(0, \infty)$
Yamabe Flow	0	0	$(0, \infty)$
Virtual Distance Circle Packing	-1	-1	$(0, \infty)$
Mixed Type	$\{-1, 0, +1\}$	$\{-1, 0, +1\}$	$(0, \infty)$

Table: Parameters for schemes.

## Definition (Entropy on a Face)

A discrete surface with  $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$  background geometry, and a circle packing metric  $(\Sigma, \gamma, \eta, \varepsilon)$ . For each triangle  $[v_i, v_j, v_k]$  with inner angle  $(\theta_i, \theta_j, \theta_k)$ , the entropy energy for the face is given by

$$E_f(u_i, u_j, u_k) = \int^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

## Definition (Entropy on a mesh)

A discrete surface with  $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$  background geometry, and a circle packing metric  $(\Sigma, \gamma, \eta, \varepsilon)$ . The discrete entropy energy for the whole mesh is defined as

$$E = \int^{(u_1, u_2, \dots, u_n)} \sum_{i=1}^n (\bar{K}_i - K_i) du_i.$$

The mesh entropy can be represented as the face energies

$$E_\sigma = \sum_{i=1}^n (\bar{K}_i - 2\pi) u_i + \sum_{f \in F} E_f.$$

# Symmetry

Suppose a triangle  $[v_i, v_j, v_k]$  is with background geometry  $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ , conformal factor  $(u_i, u_j, u_k)$ , edge length  $(l_i, l_j, l_k)$ , inner angles  $(\theta_i, \theta_j, \theta_k)$ , entropy energy is

$$E(u_i, u_j, u_k) = \int^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k. \quad (1)$$

Then the Hessian matrix is given by

$$\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = -\frac{1}{2A} L \Theta L^{-1} D, \quad (2)$$

where,  $A$  is the triangle area

$$A = \frac{1}{2} \sin \theta_i s(l_j) s(l_k), \quad (3)$$

The matrix  $L$  is

$$L = \begin{pmatrix} s(l_i) & 0 & 0 \\ 0 & s(l_j) & 0 \\ 0 & 0 & s(l_k) \end{pmatrix} \quad (4)$$

$\Theta$

$$\Theta = \begin{pmatrix} -1 & \cos \theta_k & \cos \theta_j \\ \cos \theta_k & -1 & \cos \theta_i \\ \cos \theta_j & \cos \theta_i & -1 \end{pmatrix} \quad (5)$$

matrix  $D$  is

$$D = \begin{pmatrix} 0 & \tau(i, j, k) & \tau(i, k, j) \\ \tau(j, i, k) & 0 & \tau(j, k, i) \\ \tau(k, i, j) & \tau(k, j, i) & 0 \end{pmatrix} \quad (6)$$



# Hessian Matrix

where

$$s(x) = \begin{cases} x & \mathbb{E}^2 \\ \sinh x & \mathbb{H}^2 \\ \sin x & \mathbb{S}^2 \end{cases}$$

and

$$\tau(i, j, k) = \begin{cases} \frac{1}{2}(l_i^2 + \epsilon_j \gamma_j^2 - \epsilon_k \gamma_k^2) & \mathbb{E}^2 \\ \cosh l_i \cosh^{\epsilon_j} \gamma_j - \cosh^{\epsilon_k} \gamma_k & \mathbb{H}^2 \\ \cos l_i \cos^{\epsilon_j} \gamma_j - \cos^{\epsilon_k} \gamma_k & \mathbb{S}^2 \end{cases}$$

# Geometric Interpretation

For each triangle, there is a power circle, orthogonal to three vertex circles. The distance from the power center to each edge is  $h_i, h_j, h_k$ . Then we have the geometric interpretation to the Hessian matrix: with  $\mathbb{E}^2$ ,  $\mathbb{H}^2$  and  $\mathbb{S}^2$  background geometry,

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{h_3}{l_3}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tanh h_3}{\sinh^2 l_3} \sqrt{2 \cosh^{\varepsilon_1} r_1 \cosh^{\varepsilon_2} r_2 \cosh l_3 - \cosh^{2\varepsilon_1} r_1 - \cosh^{2\varepsilon_2} r_2}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tan h_3}{\sin^2 l_3} \sqrt{-2 \cos^{\varepsilon_1} r_1 \cos^{\varepsilon_2} r_2 \cos l_3 + \cos^{2\varepsilon_1} r_1 + \cos^{2\varepsilon_2} r_2}$$