Hyperbolic Geometry

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Uniformization

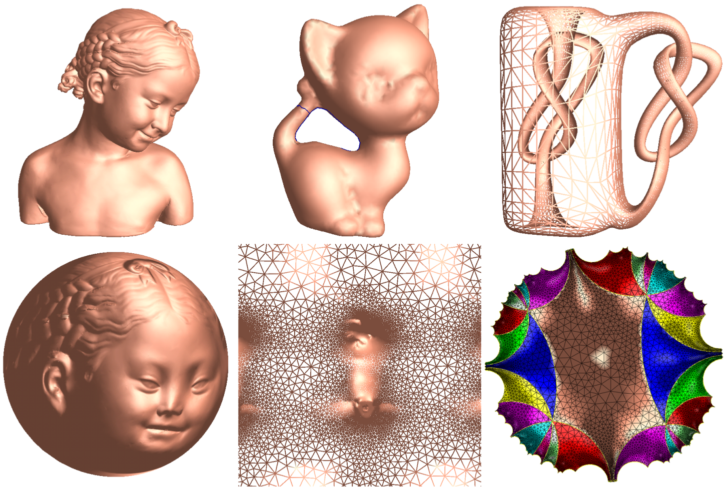


Figure: Closed surface uniformization.

Fundamental Group

Suppose $(S, \mathbf{g})$ is a closed high genus surface $g > 1$. The fundamental group is $\pi_1(S, q)$, represented as

$$\pi_1(S, q) = \langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Universal Covering Space

universal covering space of S is $\tilde{S}$, the projection map is $p : \tilde{S} \to S$. A deck transformation is an automorphism of $\tilde{S}$, $\varphi : \tilde{S} \to \tilde{S}$, $p \circ \varphi = \varphi$. All the deck transformations form the Deck transformation group $\text{Deck}\tilde{S}$.

$\varphi \in \text{Deck}(\tilde{S})$, choose a point $\tilde{q} \in \tilde{S}$, and $\tilde{\gamma} \subset \tilde{S}$ connects $\tilde{q}$ and $\varphi(\tilde{q})$. The projection $\gamma = p(\tilde{\gamma})$ is a loop on S, then we obtain an isomorphism:

$$\text{Deck}(\tilde{S}) \to \pi_1(S, q), \varphi \mapsto [\gamma]$$

Hyperbolic Structure

Uniformization

The uniformization metric is $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$, such that the $\bar{K} \equiv -1$ everywhere. Then $(\tilde{S}, \bar{\mathbf{g}})$ can be isometrically embedded on the hyperbolic plane $\mathbb{H}^2$. The On the hyperbolic plane, all the Deck transformations are isometric transformations, $Deck(\tilde{S})$ becomes the so-called Fuchsian group,

$$\text{Fuchs}(S) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g | \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\rangle.$$

The Fuchsian group generators are global conformal invariants, and form the coordinates in Teichmüller space.

Hyperbolic Plane

Definition (Poincaré Model)

The upper half complex plane $\mathbb{H}^2 = \{z \in \mathbb{C} | z = x + iy, \quad y > 0\}$ equipped with the hyperbolic metric

$$\mathbf{h} = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{y^2},$$

is the Poincaré model of hyperbolic plane. We can see the hyperbolic metric is conformal to the Euclidean metric, hence the model is angle preserving.

$$e^{2\lambda} = \frac{1}{y^2}, \quad K = -\Delta_{\mathbf{g}} \frac{1}{2} \log \frac{1}{y^2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log y = -1.$$

Hyperbolic Geometrics

Consider $\gamma(t) = (x(t), y(t)) t \in [a, b]$ the hyperbolic length of the curve is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\mathbf{h}} dt = \int_a^b \frac{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}{y(t)} dt,$$

where $\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t))$.

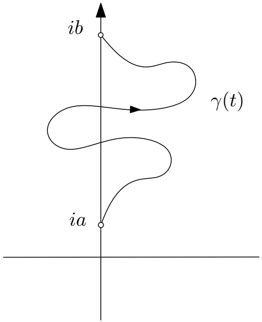


Figure: Vertical axis is a geodesic.

Hyperbolic Geodesics

Lemma

The positive imaginary axis is a geodesic

$$d(ia, ib) = \left| \ln \frac{b}{a} \right|.$$

Proof.

Construct a smooth curve $\gamma(a) = ia$, $\gamma(b) = ib$, real number $a < b$, $y(t) > 0$

$$L(\gamma) \geq \int_a^b \frac{|\dot{y}(t)|}{y(t)} dt \geq \left| \int_a^b \frac{\dot{y}(t)}{y(t)} dt \right| = \ln \frac{b}{a},$$

Equality holds, if and only if $\dot{x}(t) \equiv 0$, $\dot{y}(t) \geq 0$, equivalently $\gamma([a, b])$ is the imaginary axis, and $y(t)$ is monotonously increasing. $\square$

Isometric Transformation

Lemma

Möbius transformation $f : \mathbb{H}^2 \to \mathbb{H}^2$

$$f(z) = \frac{az+b}{cz+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}),$$

is hyperbolic isometric, where $SL(2,\mathbb{R})$ *is the special linear transformation group,* $ad - bc = 1$.

Proof.

f is the composition of the following maps

$$z \mapsto z+b, b \in \mathbb{R}; \quad z \mapsto \lambda z, \lambda \in \mathbb{R}; \quad z \mapsto -\frac{1}{z}.$$

it is obvious that $f(z) = z + b \in Iso(\mathbb{H}^2)$. □

Isometric Transformation

Proof.

Now, let $w = \lambda z$,

$$w^*(\mathbf{h}) = \frac{|dw|^2}{\Im(w)^2} = \frac{\lambda^2|dz|^2}{\lambda^2\Im(z)^2} = \frac{|dz|^2}{\Im(z)^2}.$$

hence $f(z) = \lambda z \in Iso(\mathbb{H}^2)$. Let $w = -\frac{1}{z}, dw = \frac{1}{z^2}dz, \Im(w) = \frac{1}{2i}(\frac{1}{\bar{z}} - \frac{1}{z})$,

$$w^*(\mathbf{h}) = \frac{|dw|^2}{\Im(w)^2} = \frac{\frac{1}{|z|^4}|dz|^2}{-\frac{1}{4}(\frac{1}{\bar{z}} - \frac{1}{z})^2} = \frac{\frac{1}{|z|^4}|dz|^2}{-\frac{1}{4}\frac{1}{(\bar{z}z)^2}(z-\bar{z})^2} = \frac{|dz|^2}{\Im(z)^2}.$$

hence $f(z) = -\frac{1}{z} \in Iso(\mathbb{H}^2)$. $\square$

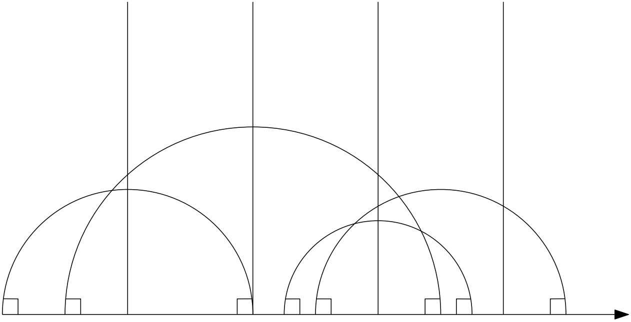


Figure: All hyperbolic geodesics are lines or circular arcs orthogonal to the real axis.

Hyperbolic Geodesics

Corollary

All hyperbolic geodesics are lines $\Re(z) = c$ *or circular arcs* $|z - a| = ra \in \mathbb{R}$ *orthogonal to the real axis.*

Proof.

imaginary axis is a geodesic, $f(z) = z + b \in Iso(\mathbb{H}^2)$, hence all vertical lines $\Re(z) = c$ are geodesics. $\square$

Hyperbolic Geodesics

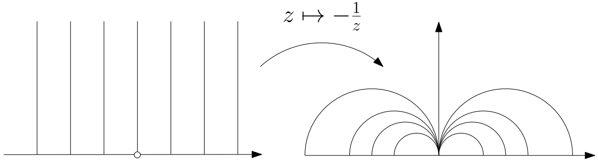


Figure: Reflection transforms vertical lines to circular arcs orthogonal to the real axis.

Isometry $z \mapsto -\frac{1}{z}$ maps a vertical line $\Re(z) = c$ to a circular arc $|w + 1/(2c)| = 1/(2|c|)$,

$$\left|\frac{-1}{c+iy} + \frac{1}{2c}\right| = \left|\frac{iy-c}{iy+c}\frac{1}{2c}\right| = \frac{1}{2|c|}.$$

composed with isometry $z \mapsto z + b$, we obtain circular arc $|z - a| = r$, $a \in \mathbb{R}$. Hence all $|z - a| = r$ are hyperbolic geodesics.

Hyperbolic Geodesic

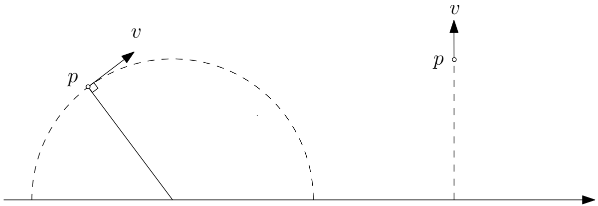


Figure: Unique geodesic through p and with tangent direction v.

We show all geodesics are vertical lines or circular arcs. Through any point p on $\mathbb{H}^2$ and with tangent direction v, there is a unique geodesic. We can construct a circular arc or vertical line τ through p with tangent direction v. By the uniqueness of geodesics, τ coincides with γ, $\gamma = \tau$. Hence all geodesics are vertical lines or circular arcs orthogonal to the real axis.

Hyperbolic Isometric Transformation Group

Lemma

Isometric transformation group on the hyperbolic plane is isomorphic to the Möbius transformation group:

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm\ Id \cong Iso(\mathbb{H}^2).$$

Proof.

We define Möbius transformation

$$\varphi_\theta(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SL(2,\mathbb{R}),$$

We have

$$\varphi_\theta(i) = i, \quad \varphi'_\theta(z) = \frac{1}{(\cos\theta - \sin\theta z)^2}, \quad \varphi'_\theta(i) = e^{i2\theta}.$$

□

Hyperbolic Transformation Group

Choose arbitrarily a $\tau \in Iso(\mathbb{H}^2)$, assume $\tau(i) = ai + b$. Construct $f \in PSL(2,\mathbb{R})$, $f(z) = az + b$, then $f(i) = \tau(i)$. Since both $\tau(z)$ and $f(z)$ are isometric transformations, hence

$$|\tau'(i)| = \Im(\tau(i)) = |f'(i)|.$$

Construct a composition map $g = f \circ \varphi_\theta$, satisfying

$$g'(i) = f'(i)\varphi'_\theta(i) = e^{i2\theta} f'(i).$$

Let $\theta = \frac{1}{2}(\arg \tau'(i) - \arg f'(i))$, then we get $g(i) = \tau(i)$ and $g'(i) = \tau'(i)$. Then we show $g(z) = \tau(z)$.

Hyperbolic Isometric Transformation Group

Construct an isometric transformation $\eta = \tau \circ g^{-1}$, then $\eta(i) = i$, and $\eta'(i) = 1$. Given a pont $z \in \mathbb{H}^2$, $z \neq i$, there is a unique geodesic γ connecting i and z, $\gamma(0) = i$, $\gamma(s) = z$, where the arc length parameter s equals to the hyperbolic distance between i and z. Isometric map η maps the geodesic γ to geodesic $\tilde{\gamma} = \eta(\gamma)$, preserving the arc length s, hence

$$\eta(z) = \eta(\gamma(s)) = \tilde{\gamma}(s). \tag{1}$$

By construction

$$\tilde{\gamma}(0) = \eta(\gamma(0)) = \eta(i) = i = \gamma(0),$$

furthermore

$$\tilde{\gamma}'(0) = \eta'(\gamma(0))\gamma'(0) = \eta'(i)\gamma'(0) = \gamma'(0).$$

hence geodesics γ and $\tilde{\gamma}$ coincide, $\gamma(s) = \tilde{\gamma}(s)$, combining Eqn.(1), we have

$$\eta(z) = \tilde{\gamma}(s) = \gamma(s) = z.$$

hence $\eta = \tau \circ g^{-1} = id$, $\tau = g$, $\tau \in SL(\mathbb{R}, 2)$.

Hyperbolic Isometric Transformation Group

By the definition of Möbius transformation, we have

$$f(z) = \frac{az+b}{cz+d}, \quad \text{corresponds to} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{and} \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix},$$

Hence Möbius transformation group is isomorphic to $PSL(\mathbb{R}, 2)$. $\square$

Cross Ration

Definition (Cross Ratio)

Suppose $a, b, c, d \in \hat{\mathbb{C}}$ are distinct points on the extended complex plane, the cross ratio is defined as

$$(a, b, c, d) = \frac{a-c}{a-d} : \frac{b-c}{b-d}.$$

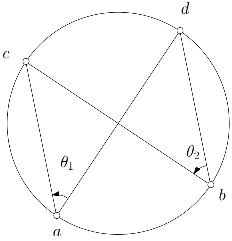


Figure: Four points are co-circular if and only if their cross ration is real.

Cross Ratio

Lemma

Assume $a, b, c, d \in \hat{\mathbb{C}}$*, the four points are co-circular, if and only if their cross ratio is real,* $(a, b, c, d) \in \mathbb{R}$*.*

Proof.

As shown in figure (6), the surficient and necessary condition for the four ponts to be co-circular is θ_1 equals to θ_2, this is equivalent to $(a, b, c, d) \in \mathbb{R}$. $\square$

Cross Ratio

Lemma (Möbius Transformation Preserving Cross Ratio)

Suppose $f \in PSL(2,\mathbb{C})$, *namely Möbius transformation*

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{C})$$

then

$$(f(a), f(b), f(c), f(d)) = (a, b, c, d). \tag{2}$$

Proof.

Consider the generators of Möbius transformation group. $f(z) = z + b$ and $f(z) = az$ preserve cross ratios. Let $f(z) = -\frac{1}{z}$, we have

$$\left(-\frac{1}{a}, -\frac{1}{b}, -\frac{1}{c}, -\frac{1}{d}\right) = \frac{a^{-1} - c^{-1}}{a^{-1} - d^{-1}} : \frac{b^{-1} - c^{-1}}{b^{-1} - d^{-1}} = \frac{a-c}{a-d}\frac{ad}{ac} : \frac{b-c}{b-d}\frac{bd}{bc} = ($$

hence all generators preserve cross ratios. □

Corollary

Möbius transformations $f \in PSL(2, \mathbb{C})$ preserve circles.

Proof.

Möbius tranformations preserve cross ratio. Four points are co-circular if and only if the cross ratio is real. □

Corollary

Suppose $z, w \in \mathbb{H}^2$ the hyperbolic geodesic through z, w intersects the real axis at infinity z', w', then the hyperbolic distance between z and w is the logarithm of the cross ratio,

$$d(z, w) = \ln(z, w, w', z').$$

Cross Ratio

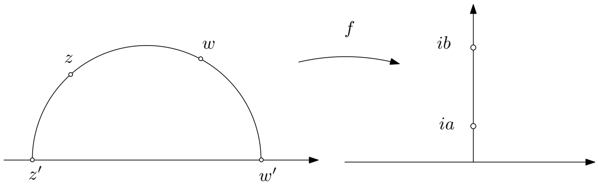


Figure: Cross ratio and the hyperbolic distance.

We choose a Möbius transformation $f_k \in PSL(2,\mathbb{R})$ as follows,

$$f(z) = \frac{z - z'}{z - w'}$$

then $f(z') = 0$, and $f(w') = \infty$, the geodesic is mapped to the positive imaginary axis. Assume $f(z) = ia$, $f(w) = ib$, $a < b$. Since f is hyperbolic isometry, we have

$$d(z,w) = d(ia, ib) = \ln \frac{b}{a} = \ln(ia, ib, \infty, 0) = \ln(z, w, w', z').$$

Hyperbolic Triangle

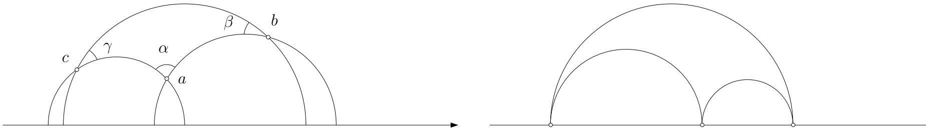


Figure: Hyperbolic triangle.

Given three distinct points a, b, c on the hyperbolic plane $\mathbb{H}^2$ their hyperbolic convex hull is a hyperbolic triangle, all three edges are hyperbolic lines. Assume the inner angles are α, β, γ, by Gauss-Bonnet,

$$\int_\Delta K dA_{\mathbf{h}} + \int_\partial \Delta k_g ds + (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi\chi(\Delta),$$

$\chi(\Delta) = 1$, then we have the area of the hyperbolic triangle,

$$A(\Delta) = \pi - \alpha - \beta - \gamma.$$

Hyperbolic Triangle

Definition (Hyperbolic Ideal Triangle)

If all the vertices of the triangle are infinity, then the triangle is called an ideal hyperbolic triangle, three inner angles are 0's, the area is π.

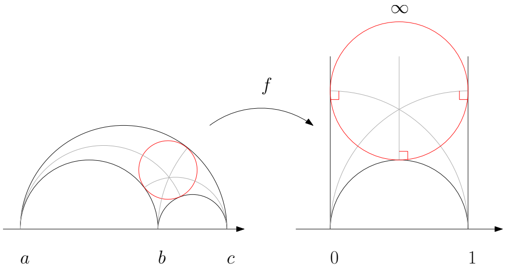


Figure: All hyperbolic ideal triangles are isometric.

Hyperbolic Ideal Triangle

For any hyperbolic ideal triangle with vertices (a, b, c), we construct a Möbius transformation

$$f(z) = \frac{z-a}{z-c}\frac{b-c}{b-a},$$

which maps $\{a, b, c\}$ to $\{0, 1, \infty\}$. All hyperbolic ideal triangles are isometric to the canonical ideal triangle $(0, 1, \infty)$, therefore all hyperbolic ideal triangles are isometric.
The three heights of an ideal triangle intersect at a single point, which is also the center of the inner circle of the triangle. The inner circle intersects the three edges at the perpendicular feet, which are called the middle points of the edges.

Poincaré Disk

Möbius transformation $\varphi : \mathbb{H} \to \mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$,

$$\varphi(z) = \frac{z - i}{z + i},$$

φ maps $\{0, 1, \infty\}$ to $\{-1, -i, 1\}$, and the upper half plane to the unit disk. The hyperbolic metric on $\mathbb{D}$ is:

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

The isometric transformations on Poincaré disk are Möbius transforamtions:

$$z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

The geodesics are diameters or circular arcs perpendicular to the unit circle.

Hyperbolic Geodesics on Poincaré's Disk

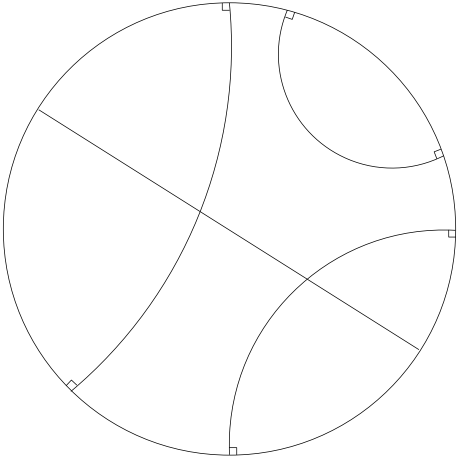


Figure: Hyperbolic geodesics on Poincaré disk.

Hyperbolic Geodesics on Poincaré's Disk

Lemma (Geodesic Circle)

The geodesic circles on Poincare's disk model or upper half plane model coincide with Euclidean circles.

Proof.

Given a hyperbolic circle $d(z,a) = r$ in $\mathbb{D}$, we use Möbius transformation

$$f(z) = \frac{z-a}{1-\bar{a}z},$$

maps the circle center to the origin, and obtain a hyperbolic circle $d(z,0) = r$. By symmetry, the hyperbolic circle centered at the origin is also a Euclidean circle,

$$|z| = \frac{e^r - 1}{e^r + 1} = \tanh\frac{r}{2}.$$

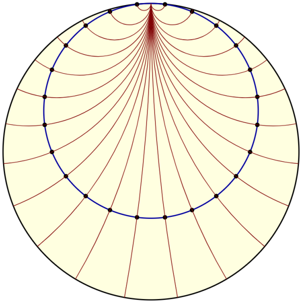
$\square$

Hyperbolic Geodesics on Poincaré's Disk

Proof.

We use f^{-1} map the Euclidean circle back to the original position. Since Möbius transformation preserves circles, hence $d(z, a) = r$ is a Euclidean circle. Since the upper half plane model and the unit disk model differ by a Möbius transformation, hence hyperbolic circle in upper half plane model is also a Euclidean model. $\square$

Hyperbolic Horocircle



Definition (Horocircle)

A horocircle on the hyperbolic plane is a curve γ, such that all the hyperbolic geodesics orthogonal to γ converge to the same direction asymptotically.

Hyperbolic Sine Law and Cosine Law

Theorem

Given a hyperbolic triangle, the cosine law is given by:

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(A).$$

and

$$\cos(B) = -\cos(C) + \sin(A)\sin(C)\cosh(b).$$

the sine law is

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$

Hyperbolic Structure

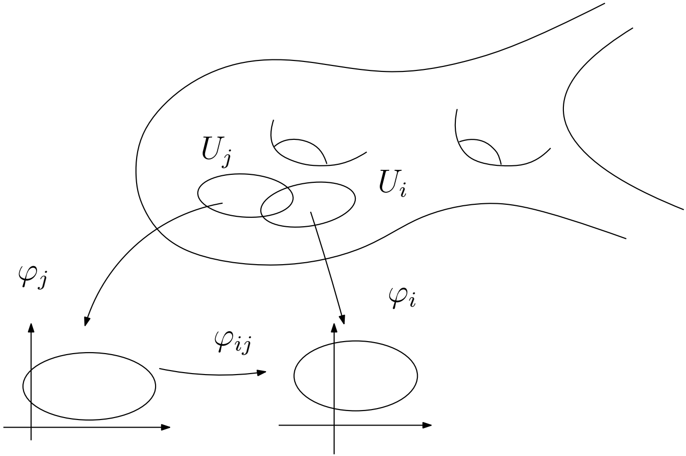


Figure: Surface hyperbolic structure.

Definition (Hyperbolic Structure)

A hyperbolic structure of a surface is an atlas, $\{(U_i, \varphi_i) | i \in I\}$, such that

1. $\Sigma = \bigcup_i U_i$
2. $\varphi_i : U_i \to \mathbb{H}^2$ is a local chart,
3. local coordinate transformation is $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} = g \Big|_{\varphi_j(U_i \cap U_j)}$, $g \in Iso(\mathbb{H}^2)$.

The hyperbolic structure is complete, if any geodesic can be extended infinitely. The geodesic and angle on (Σ, φ) can be defined using the local coordinates.

Hyperbolic Structure

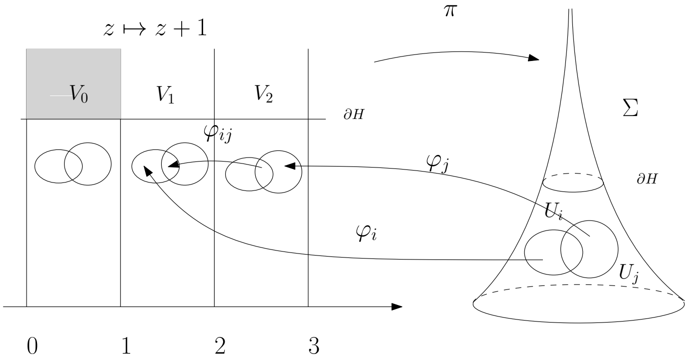


Figure: Cusps on a hyperbolic surface.

Cusp on Hyperbolic Surface

Let $\gamma(z) = z + 1$,γ generates the transformation group $\Gamma = \langle \gamma^n \rangle$, $\Gamma \subset Iso(\mathbb{H}^2)$. The quotient space $\Sigma = \mathbb{H}^2/\Gamma$ has a hyperbolic cusp, $\pi : \mathbb{H}^2 \to \Sigma$ is the projection map, local chart $(\varphi_i, \mathbb{H}^2)$, $\varphi_i = (\pi\, |_{V_i})^{-1} \varphi_j = (\pi\, |_{V_j})^{-1}$, local coordinates transformation $\varphi_{ij} \in \langle \gamma \rangle$. We place a horocircle at the cusp H, the horocircle includes a neighborhood of the cusp, the intersection between the horocircle and Σ is ∂H. Suppose ∂H is represented as $\Im(z) = y$ in $\mathbb{H}^2$, then the length of the intersection is

$$L(\partial H) = \int_0^1 \frac{dx}{y} = \frac{1}{y}.$$

the area of the neighborhood is

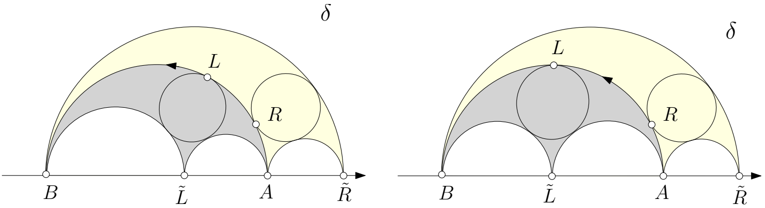
$$Area(H) = \int_0^1 \int_y^\infty \frac{dxdy}{y^2} = \frac{1}{y}.$$

Ideal Quadrilateral

Definition (Ideal Quadrilateral)

Given 4 infinite point $v_1, v_2, v_3, v_4 \in \partial\mathbb{H}^2$, its convex hull is called a *hyperbolic ideal quadrilateral*.

Different ideal quadrilaterals are not isometric.



Ideal Quadrilateral

Definition (marked ideal quadrilateral)

A marked ideal quadrilateral is the union of two oriented ideal quadrilateral glued isometrically along the common edge.

Every ideal triangle has an inner circle, which intersects the triangle at three perpendicular feet. The oriented distance between the two perpendicular feet on the diagonal of the ideal quadrilateral is called the shear coordinate of the quadrilateral.

Definition (Shear Coordinate)

Given a marked ideal quadrilateral δ, Thurston's Shear Coordinates $d(\delta)$ equals to the oriented distance along the diagonal from L to R.

Ideal Quadrilateral

Lemma

Suppose $\delta = [A, \tilde{R}, B, \tilde{L}]$, *then*

$$d(\delta) = \ln -(A, B, \tilde{R}, \tilde{L}).$$

Proof.

By a Möbius transformation, $\{A, B, \tilde{L}, \tilde{R}\}$ are mapped to $\{0, \infty, -1, t > 0\}$,

$$(A, B, \tilde{R}, \tilde{L}) = (0, \infty, t, -1) = \frac{0-t}{0+1} : \frac{\infty - t}{\infty + 1} = -t.$$

then $L = i$, $R = it$, hence $d(\delta) = \ln(t)$. □

Thurston's Shear Coordinates

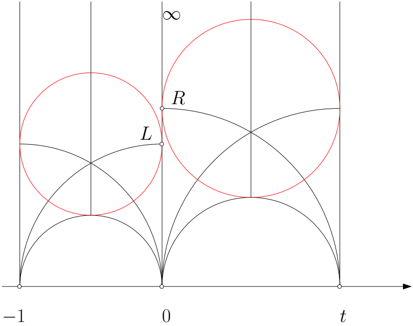


Figure: Thurston's shear coordinates of an ideal quadrilateral.

Assume a genus g surface with n punctures, $\Sigma = \Sigma_g - \{v_1, v_2, \ldots, v_n\}$, $n \geq 1$, $\chi(\Sigma) < 0$, $(\Sigma, \mathcal{T})$ is an ideal triangulation,

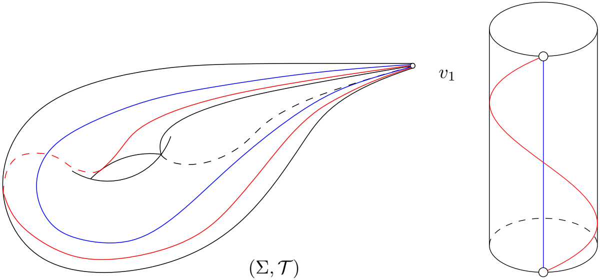


Figure: Ideal triangulation of a torus with one puncture.

Hyperbolic Structure

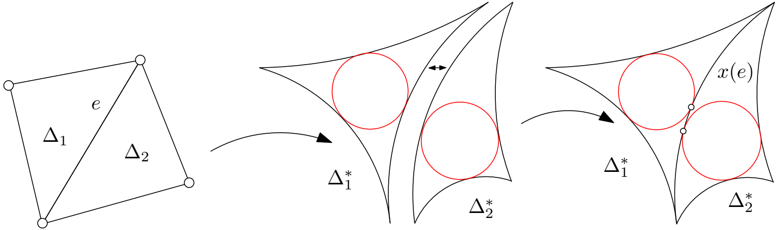


Figure: Glue ideal triangles, with shear coordinates $x(e)$.

Given any function $x : E(\mathcal{T}) \to \mathbb{R}$, denoted as $x \in \mathbb{R}^{E(\mathcal{T})}$, we can construct a hyperbolic structure $\pi(X)$ of Σ by the following construction,

1. For every triangle $\Delta \in \mathcal{T}$, construct an ideal hyperbolic triangle, $\Delta \to \Delta^*$;
2. For each edge $e \in E(\mathcal{T})$, isometrically glue two ideal triangles Δ_1^* and Δ_2^* along e, such that the shear coordinates on e equals to $x(e)$.

Glue Pattern

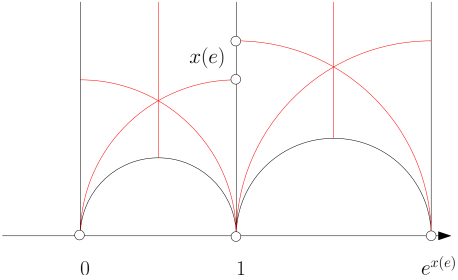


Figure: Gluing ideal triangles with shear coordinates $x(e)$.

Lemma

$\pi(x)$ is a complete hyperbolic metric with finite area, if and only if for any vertex $v \in \{v_1, v_2, \cdots, v_n\}$,

$$\sum_{e \sim v} x(e) = 0. \tag{3}$$

Complete Metric Condition

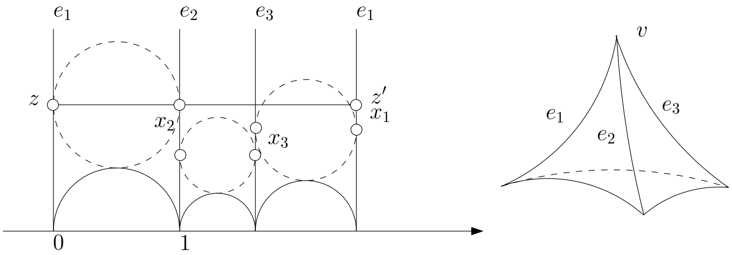


Figure: Condition for complete metric with finite area.

Proof.

Assume v is ∞, then z is equivalent to z', $z \sim z'$. By construction,

$$\Im(z')/\Im(z) = e^{x_1+x_2+x_3}.$$

v is a cusp, if and only if the line connecting z and z' is a horocircle. Hence $x_1 + x_2 + x_3 = 0$. $\square$

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Teichmüller Space of Complete Hyperbolic Metrics)

The Teichmüller space of complete hyperbolic metrics with finite areas on (Σ, V) is defined as

$$T(\Sigma) = \frac{\{\text{complete hyperbolic metrics with finite area on } (\Sigma, V)\}}{\{\text{isometric transformations } \sim \text{ identity, preserving cusps}\}}.$$

We have the structural decomposition:

$$T_D(\Sigma) = T(\Sigma) \times \mathbb{R}^n_{>0} \tag{4}$$

where $T(\Sigma)$ is the Teichmüller space of complete hyperbolic metrics with finite area, $\mathbb{R}^n_{>0}$ representing the lengths of ∂H_i (decorations).

Teichmüller Space of Complete Hyperbolic Metric

Theorem (Thurston)

Define linear subspace:

$$\mathbb{R}^E_P = \left\{ x \in \mathbb{R}^E \middle| \forall v \in V, \sum_{v \sim e} x(e) = 0 \right\}$$

The mapping

$$\Phi_{\mathcal{T}} : \mathbb{R}^E_P \to T(\Sigma), x \mapsto [\pi(x)]$$

is a bijection. The hyperbolic metric $\Phi_{\mathcal{T}}(x)$ has the Thurston's shear coordinates $x(e)$ under the triangulation $\mathcal{T}$. $T(\Sigma)$ is the Teichmüller space of all complete hyperboic metrics with finite area on (Σ, V).

Teichmüller Space of Hyperbolic Metrics

Theorem

Suppose ideal triangulations $\mathcal{T}$ and $\mathcal{T}'$ differ by an edge swap, then coordinates transformation

$$\Phi_{\mathcal{T}'}^{-1} \circ \Phi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to \mathbb{R}^{E(\mathcal{T}')}$$

has the following formulation:

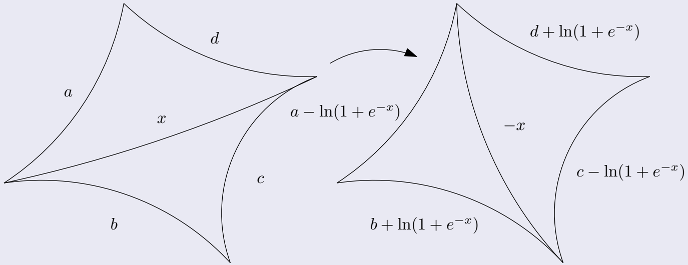


Figure: *Edge swap induces coordinates transformation.*

Teichmüller Space of Hyperbolic Metric

Corollary

Teichmüller space $T(\Sigma)$ is a real analytic manifold, which is diffeomorphic to $\mathbb{R}^{6g-6+2n}$.

Proof.

The dimension of $T(\Sigma)$ equals to the number of edges minus the number of vertices. Given a closed surface Σ with genus g, $V(\mathcal{T}) + F(\mathcal{T}) - E(\mathcal{T}) = 2 - 2g$ and $3F(\mathcal{T}) = 2E(\mathcal{T})$, we obtain $E(\mathcal{T}) = 6g - 6 + 3n$. Hence the dimension of $T(\Sigma)$ equals to $6g - 6 + 2n$, where $V(T) = n$. $\square$

Penner's λ-length

Given a surface with punctures $(\Sigma, \mathbf{d})$, where $\mathbf{d}$ is a complete hyperbolic metric with finite area. A horoball is a sub-surface, isometric to the quotient space $\{\Im(z) > c\}/(z \sim z+1)$, which is a neighborhood of the cusp of Σ, denoted as H. Then we have

$$\text{area}(H) = \text{length}(\partial H).$$

Let $H_1, H_2, \dots, H_n$ be the horoballs at the vertices (cusps),

$$\Sigma - \bigcup_{i=1}^{n} H_i$$

is a compact surface, every boundary ∂H_i is a horocircle.

Penner's λ-length

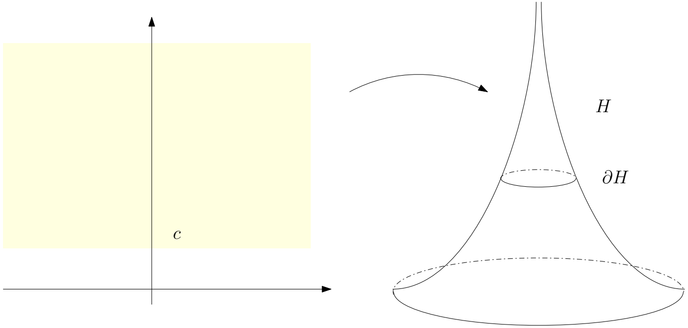


Figure: Horoball $\{\Im(z) > c\}/(z \sim z+1)$

Decorated Ideal Triangle

- A decorated ideal triangle τ is an ideal triangle, with infinite vertices $v_1, v_2, v_3 \in \partial\mathbb{H}^2$. Every vertex v_i is associated with a horoball H_i;
- the angle α_i at the vertex v_i is the length of the intersection between the boundary of horoball ∂H_i and τ;
- each infinite edges e_i is against v_i, $i = 1, 2, 3$, the oriented hyperbolic length of e_i is l_i. If $H_j \cap H_k = \emptyset$ then l_i is positive, otherwise, if $H_j \cap H_j \neq \emptyset$, then l_i is negative.
- Penner's λ-length L_i is defined as:

$$L_i := e^{\frac{1}{2} l_i}. \tag{5}$$

Decorated Ideal Triangle

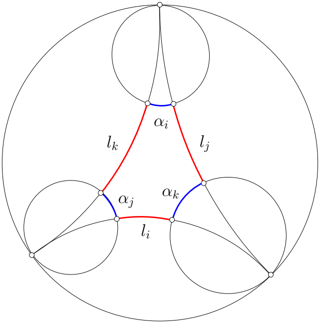


Figure: Decorated ideal hyperbolic triangle, left frame $l_i > 0$, right frame $l_i < 0$.

Cosine Law of Decorated Ideal Triangle

Theorem (Cosine Law for Decorated Ideal Hyperbolic Triangle)

Given a decorated hyperbolic ideal triangle,

1. *cosine law:*
$$\alpha_i = e^{\frac{1}{2}(l_i - l_j - l_k)} = \frac{L_i}{L_j L_k}.$$
2. *the distance from the horocircle ∂H_i to the perpendicular foot p_j (or p_k) equals to $-\ln \alpha_i$.*
3. *Given arbitrary $l_1, l_2, l_3 \in \mathbb{R}$, there exists a unique decorated hyperbolic triangle τ with oriented hyperbolic lengths $\{l_1, l_2, l_3\}$.*

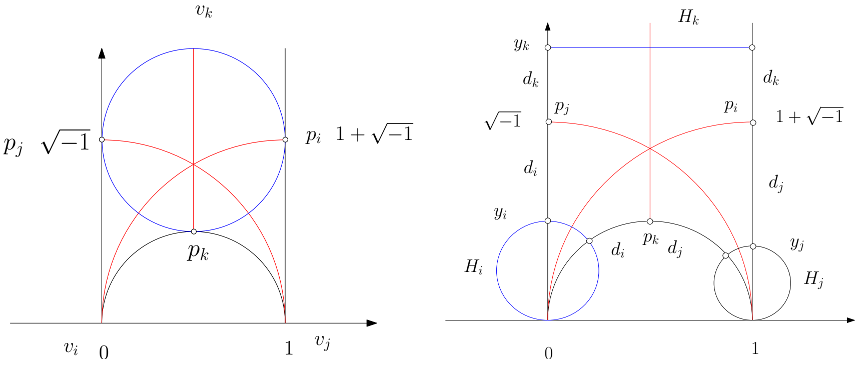


Figure: Decorated ideal hyperbolic triangle Cosine law.

Cosine Law of Decorated Ideal Triangle

Proof.

Given any Penner's λ-lengths $\{L_i, L_j, L_k\}$, the oriented hyperbolic lengths are

$$\{l_i, l_j, l_k\} = \{2\ln L_i, 2\ln L_j, 2\ln L_k\},$$

compute $\{d_i, d_j, d_k\}$,

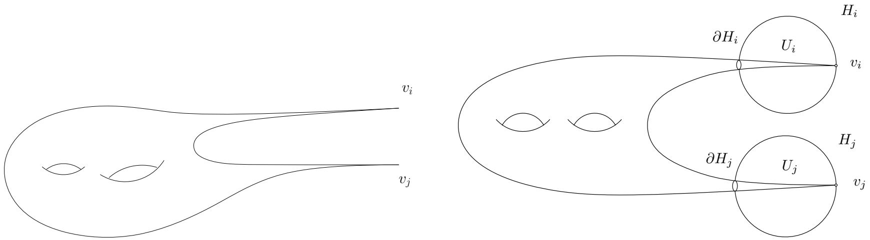
$$\begin{aligned} d_i &= (l_j + l_k - l_i)/2 \\ d_j &= (l_k + l_i - l_j)/2 \\ d_k &= (l_i + l_j - l_k)/2 \end{aligned}$$

then compute

$$\{y_i, y_j, y_k\} = \{e^{-d_i}, e^{-d_j}, e^{+d_k}\}.$$

□

Decorated Hyperbolic Metric



(a) complete metric with finite area (b) decorated hyperbolic metric

Figure: Hyperbolic metrics

Decorated Hyperbolic Metric

Definition (Decorated Hyperbolic Metric)

A decorated hyperbolic metric on a surface Σ is a tuple $(\mathbf{d}, \mathbf{w})$

1. $\mathbf{d}$ is a complete hyperbolic with finite area;
2. Every cusp v_i is associated with a horoball $H_i$centered at v_i, the length of the boundary ∂H_i is w_i, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_{>0}$.

Definition (Teichmüller Space of Decorated Hyperbolic Metrics)

The Teichmüller space of decorated hyperbolic metrics on Σ is defined as

$$T_D = \frac{\{[(\mathbf{d},\mathbf{w})] | (\mathbf{d},\mathbf{w})\ \Sigma \text{ decorated hyperbolic metric}\}}{\{\text{isometric transformations } \sim \text{ identity, preserving horoballs}\}}.$$

We have the structural decomposition:

$$T_D(\Sigma) = T(\Sigma) \times \mathbb{R}^n_{>0} \tag{6}$$

where $T(\Sigma)$ is the Teichmüller space of complete hyperbolic metrics with finite area, $\mathbb{R}^n_{>0}$ representing the lengths of ∂H_i (decorations).

Euclidean Metric to Decorated Hyperbolic Metric

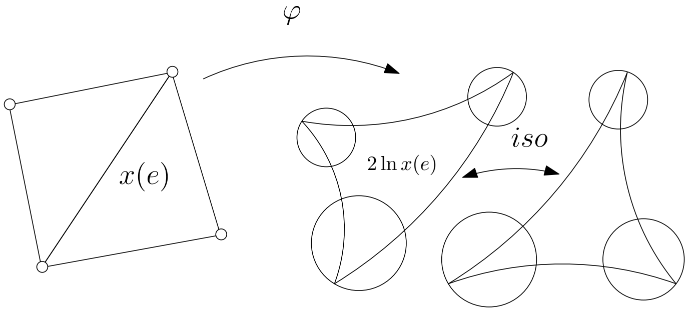


Figure: Conversion from a Euclidean metric to a decorated hyperbolic metric.

Euclidean Metric to Decorated Hyperbolic Metric

Suppose Σ is a surface with polyhedral metric and a triangulation $\mathcal{T}$, the Euclidean edge length function is $x \in \mathbb{R}_{>0}^{E(\mathcal{T})}$. Construct a decorated hyperbolic metric $\varphi(x) \in T_D(\Sigma)$:

1. convert each Euclidean face $\Delta \in F(\mathcal{T})$ to a decorated hyperbolic triangle. In each triangle $\{e_i, e_j, e_k\}$, treat Euclidean edge lengths $\{x(e_i), x(e_j), x(e_k)\}$ as Penner's λ-lengths to obtain hyperbolic edge lengths

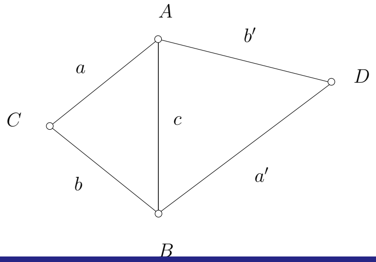
$$\Phi_{\mathcal{T}} : \{x(e_i), x(e_j), x(e_k)\} \mapsto \{2\ln x(e_i), 2\ln x(e_j), 2\ln x(e_k)\}.$$

2. isometrically glue the decorated hyperbolic triangles along the common edges, preserving the decorations.

Theorem

Fix a triangulation $\mathcal{T}$ of Σ, $\Phi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to T_D(\Sigma)$ is a topological homeomorphism.

Cross Ratio



Definition (Length Cross Ratio)

Given a triangulation for a closed polyhedral surface $(\Sigma, \mathbf{d}, \mathcal{T})$, for each pair of adjacent faces $\{A, C, B\}$ and $\{A, B, D\}$, the common edge is $\{A, B\}$, then the length cross ratio is defined as:

$$\mathrm{Cr}(\{A, B\}) := \frac{aa'}{bb'},$$

where a, a', b, b' are the lengths of edges $\{A, C\}, \{B, D\}, \{B, C\}, \{A, D\}$ under the polyhedral metric.

Shear Coordinates

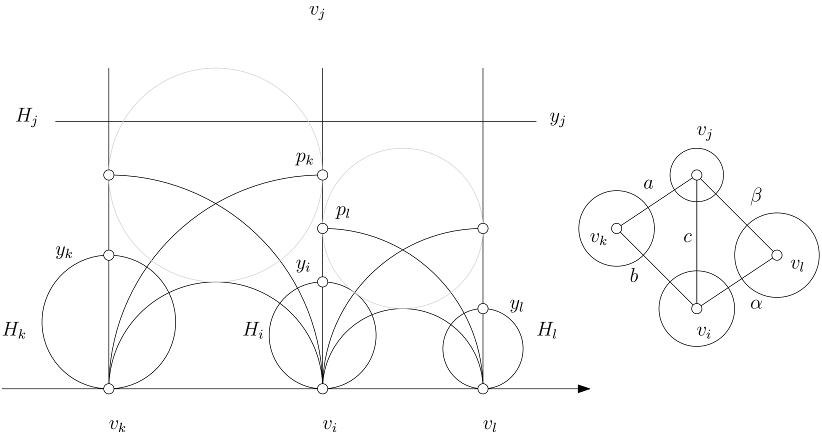


Figure: Conversion from Euclidean metric to decorated hyperbolic metric, shear coordinates.

Shear Coordinates

Theorem (Pennerλ-length to Shear Coordinates)

Given a triangulated, closed polyhedral surface $(\Sigma, \mathbf{d}, \mathcal{T})$, with Euclidean metric $\mathbf{d}$, is converted to a decorated hyperbolic metric $(\Sigma, \rho, \mathcal{T})$ by $\Phi_{\mathcal{T}}$, the shear coordinates of each edge under the hyperbolic metric ρ equals to the logarithm of the length cross ratio $-\ln Cr(e)$ under the Euclidean metric $\mathbf{d}$.

Ptolemy Equation

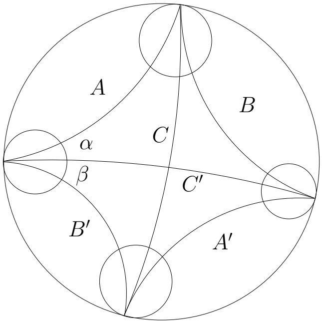


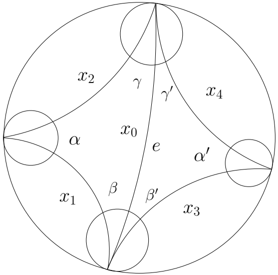
Figure: Penner Ptolemy equation.

Corollary (Penner Ptolemy)

Let A, A', B, B', C, C' be the λ-lengths of a decorated ideal quadrilateral, then

$$CC' = AA' + BB'.$$

Decorated Hyperbolic Delaunay Triangulation



Definition (Decorated Hyperbolic Delaunay Triangulation)

Suppose $(S, V, \mathcal{T}, \mathbf{d}, \mathbf{w})$ is a triangulated surface with a decorated hyperbolic metric. Suppose each $e \in E(\mathcal{T})$ satisfies

$$\alpha + \alpha' \leq \beta + \beta' + \gamma + \gamma', \tag{7}$$

then we say $\mathcal{T}$ is Penn Delaunay.

Euclidean Delaunay vs. Euclidean Delaunay

Lemma

A triangulation $\mathcal{T}$ is Delaunay under a decorated hyperbolic, if and only if for every edge $e \in E(\mathcal{T})$ satisfies

$$\frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0. \tag{8}$$

Corollary

Given a triangulated marked surface $(S, V, \mathcal{T})$, with a polyhedral metric $x : E(\mathcal{T}) \to \mathbb{R}^+$, the decorated hyperbolic metric is $\Phi_{\mathcal{T}}(x)$. Then $\mathcal{T}$ is Delaunay under the polyhedral metric x, if and only if $\mathcal{T}$ is Delaunay under the decorated hyperbolic metric $\Phi_{\mathcal{T}}(x)$.