Simplicial Homology and Cohomology Groups

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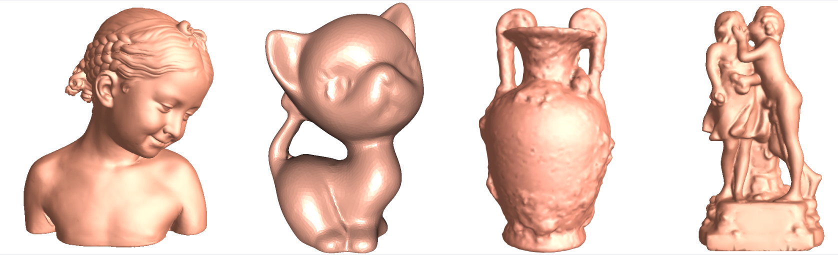
Homology and Cohomology Group

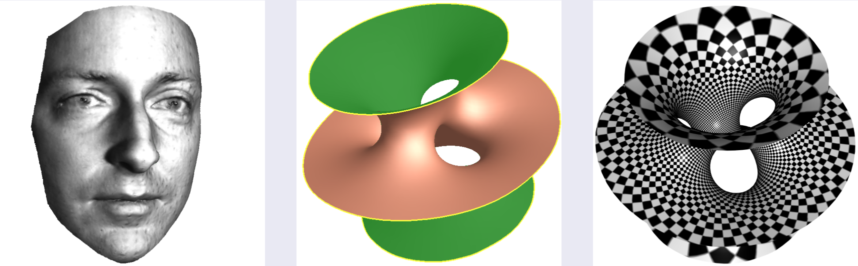
Topology of Surfaces - Closed Surfaces



genus 0 genus 1 genus 2 genus 3

Figure: Surface topological classification

Topology of Surfaces - Surfaces with boundaries



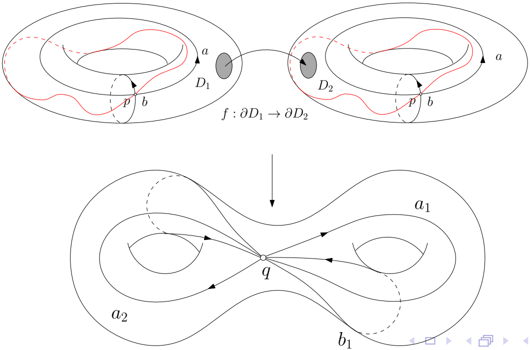
(0,1) (1,3) (1,3)

Figure: Topological classification for surfaces with boundaries (g, b).

Definition (connected Sum)

The connected sum $S_1 \oplus S_2$ is formed by deleting the interior of disks D_i and attaching the resulting punctured surfaces $S_i - D_i$ to each other by a homeomorphism $h : \partial D_1 \to \partial D_2$, so

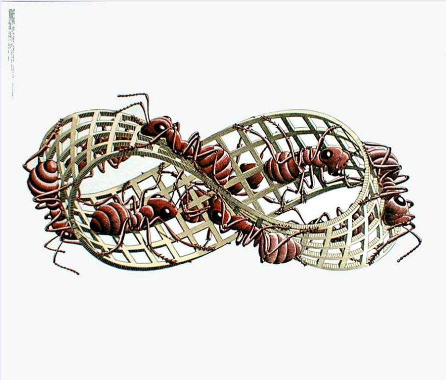
$$S_1 \oplus S_2 = (S_1 - D_2) \cup_h (S_2 - D_2).$$



Connected Sum

A Genus eight Surface, constructed by connected sum.

M.c. Escher



Möbius band.

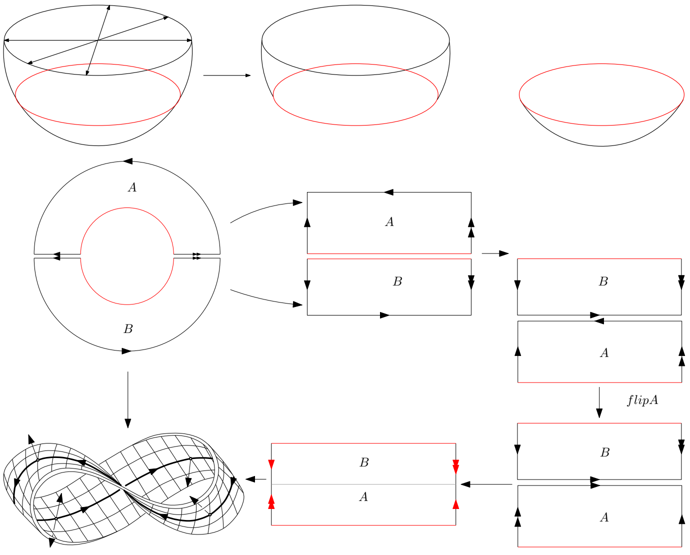
Projective Plane

Definition (Projective Plane)

All straight lines through the origin in $\mathbb{R}^3$ form a two dimensional manifold, which is called the projective plane RP^2.

A projective plane can be obtained by identifying two antipodal points on the unit sphere. A projective plane with a hole is called a crosscap. $\pi_1(RP^2) = \{\gamma, e\}$.

Projective Plane



Surface Topology

Theorem (surface Topology)

Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a sphere with a finite number of disjoint discs removed and with cross caps glued in their places. The sphere and connected sums of tori are orientable surfaces, whereas surfaces with crosscaps are unorientable.

Any closed surface is the connected sum

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_g,$$

if S is orientable, then S_i is a torus. If S is non-orientable, then S_i is a projective plane.

Triangular mesh

Definition (triangular mesh)

A triangular mesh is a surface Σ with a triangulation T,

1. Each face is counter clockwisely oriented with respect to the normal of the surface.
2. Each edge has two opposite half-edges.

Simplicial Complex

Definition (Simplicial Complex)

Suppose $k+1$ points in the general positions in $\mathbb{R}^n$, $v_0, v_1, \cdots, v_k$, the standard simplex $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \cdots, v_k$ as the vertices of the simplex σ.

Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a facet of σ.

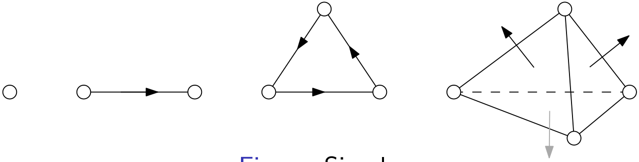


Figure: Simplex

Simplicial Complex

Definition (Simplicial complex)

A simplicial complex Σ is a union of simplices, such that

1. If a simplex σ belongs to Σ, then all its facets also belongs to Σ.
2. If $\sigma_1, \sigma_2 \subset \Sigma$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then their intersection is also a common facet.

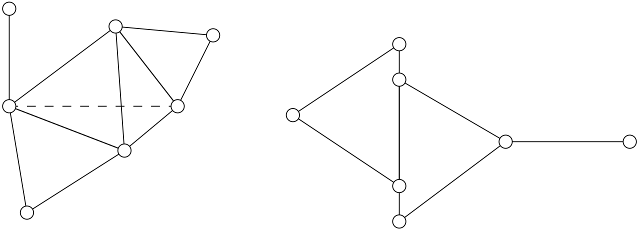


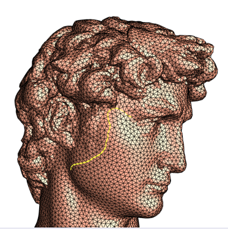
Figure: Simplicial complex.

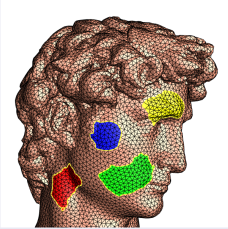
Chain Space

Definition (Chain Space)

A k chain is a linear combination of all k-simplicies in Σ, $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}$. The k dimensional chain space is the linear space formed by all k-chains, denoted as $C_k(\Sigma, \mathbb{Z})$.

A curve on the mesh is a 1-chain, a surface patch is a 2-chain.



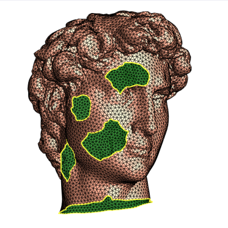


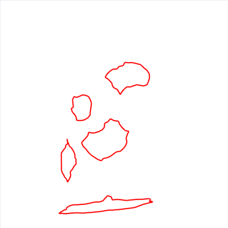
Definition (Boundary Operator)

The n-th dimensional boundary operator $\partial_n : C_n \to C_{n-1}$ is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \cdots, v_n] = \sum_i (-1)^i [v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n].$$

Boundary operator extracts the boundary of a chain.





Boundary Operator

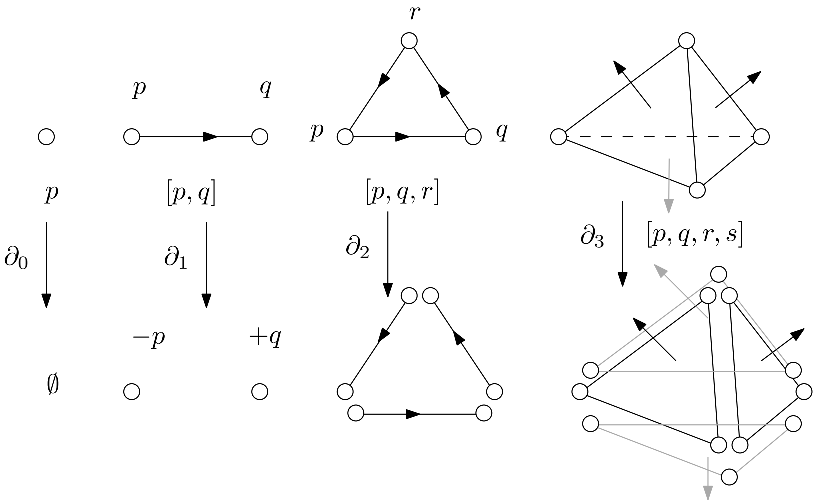


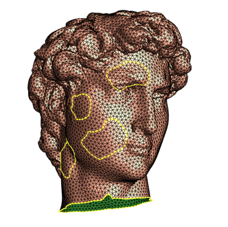
Figure: Boundary operator.

Closed Chains

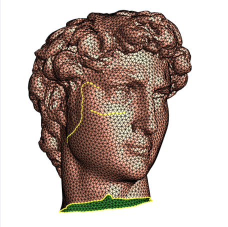
Definition (closed chain)

A k-chain $\gamma \in C_k(\sigma)$ is called a closed k-chain, if $\partial_k \gamma = 0$.

A closed 1-chain is a loop. A non-closed 1-chain has boundary vertices.



closed 1-chain

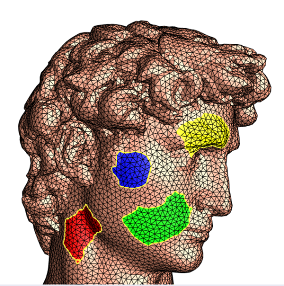


open 1-chain

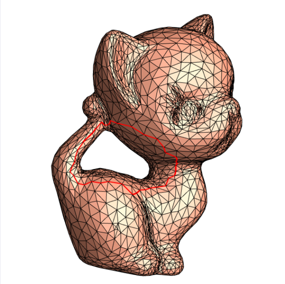
Exact Chains

Definition (Exact Chain)

A k-chain $\gamma \in C_k(\sigma)$ is called an exact k-chain, if there exists a $(k+1)$ chain σ, such that $\partial_{k+1}\sigma = \gamma$.



exact 1-chain



closed, non-exact 1-chain

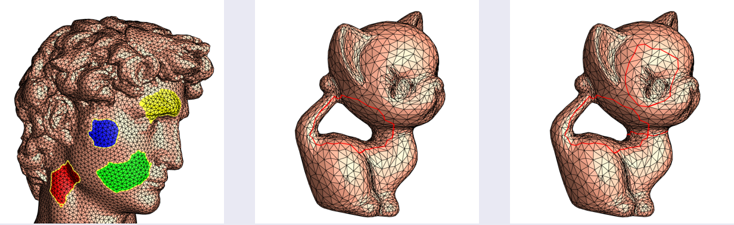
Boundary of Boundary

Theorem (Boundary of Boundary)

The boundary of a boundary is empty

$$\partial_k \circ \partial_{k+1} \equiv \emptyset.$$

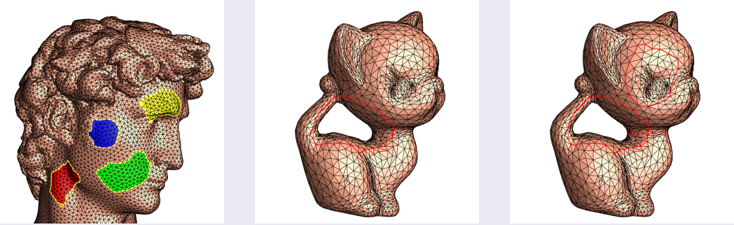
namely, exact chains are closed. But the reverse is not true.



Homology

The difference between the closed chains and the exact chains indicates the topology of the surfaces.

1. Any closed 1-chain on genus zero surface is exact.
2. On tori, some closed 1-chains are not exact.



Homology Group

Closed k-chains form the kernel space of the boundary operator ∂_k. Exact k-chains form the image space of ∂_{k+1}.

Definition (Homology Group)

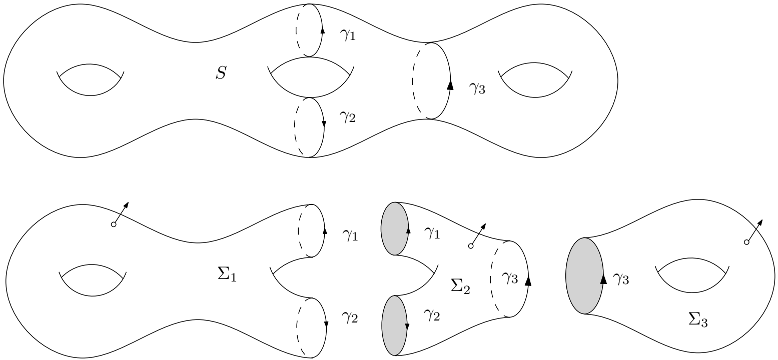
The k dimensional homology group $H_k(\Sigma, \mathbb{Z})$ is the quotient space of $ker\partial_k$ and the image space of $img\partial_{k+1}$.

$$H_k(\Sigma, \mathbb{Z}) = \frac{ker\partial_k}{img\partial_{k+1}}.$$

Two k-chains γ_1,γ_2 are homologous, if they boundary a $(k+1)$-chain σ,

$$\gamma_1 - \gamma_2 = \partial_{k+1}\sigma.$$

Homological Classes



$$\partial\Sigma_1 = \gamma_1 - \gamma_2, \quad \partial\Sigma_2 = \gamma_3 - \gamma_1 + \gamma_2, \quad \partial\Sigma_3 = -\gamma_3.$$

γ_1 and γ_2 are not homotopic but homological; γ_3 is not homotopic to e, but homological to 0; γ_3 is homological to $\gamma_1 - \gamma_2$.

Homology vs. Homotopy

Abelianization

The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

$$H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)].$$

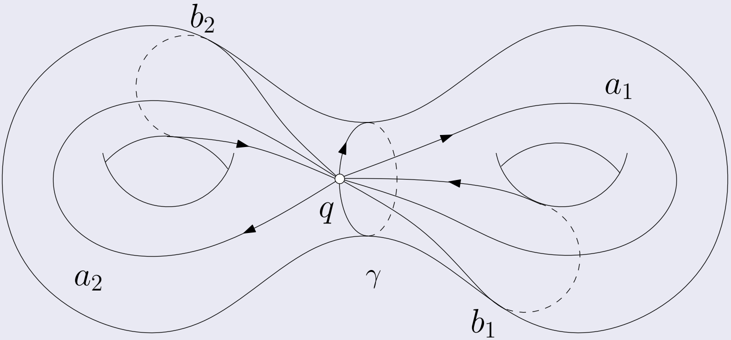
where $[\pi_1(\Sigma), \pi_1(\Sigma)]$ is the commutator of π_1,

$$[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}.$$

Fundamental group encodes more information than homology group, but more difficult to compute.

Homology vs. Homotopy

Homotopy group is non-abelian, which encodes more information than homology group.



- in homotopy group $\pi_1(S, q)$, $\gamma \sim [a, b]$,
- in homology group $H_1(S, \mathbb{Z})$, $\gamma \sim 0$.

Poincaré Duality

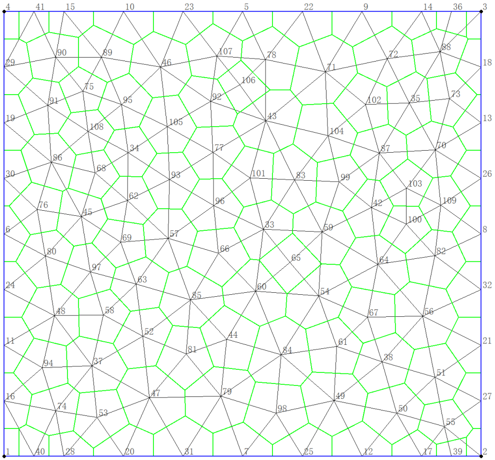
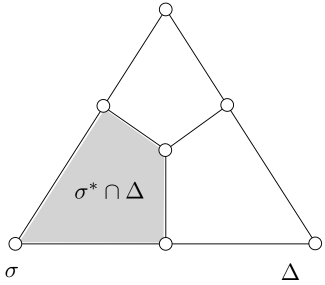


Figure: Poincaré Duality.

Poincaré Duality

Given a triangulated manifold T, there is a corresponding dual polyhedral decomposition T^*, which is a cell decomposition of the manifold such that the k-cells of T^* are in bijective correspondence with the $(n-k)$-cells of T.

Let σ be a simplex of T. Let Δ be a top-dimensional simplex of T containing σ, so we can think of σ as a subset of the vertices of Δ. Define the dual cell σ^* corresponding to σ so that $\Delta \cap \sigma^*$ is the convex hull in Δ of the barycentres of all subsets of the vertices of Δ that contain σ.



Theorem

Suppose M is a n dimensional closed manifold, then $H_k(M,\mathbb{Z}) \cong H_{n-k}(M,\mathbb{Z})$.

Proof.

The intersection map $C_k(T) \times C_{n-K}(T^*) \to \mathbb{Z}$ gives an isomorphism $C_k(T) \to C^{n-k}(T^*)$. $\square$

Theorem

Suppose M is a genus g closed surface, then $H_0(M,\mathbb{Z}) \cong \mathbb{Z}$, $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H_2(M,\mathbb{Z}) \cong \mathbb{Z}$.

If $H_0(M,\mathbb{Z}) = \mathbb{Z}^k$, then M has k connected components.

Each boundary operator: $\partial_k : C_k \to C_{k-1}$ is a linear map between linear spaces C_k and C_{k-1}, therefore it can be represented as a integer matrix. Suppose there are n_k k-simplexes of Σ, $\{\sigma_1^k, \sigma_2^k, \dots, \sigma_{n_k}^k\}$.

$$C_k = \left\{ \sum_{i=1}^{n_k} \lambda_i \sigma_i^k \right\}.$$

Boundary Matrix

The boundary matrix is defined as: $\partial_k = ([\sigma_i^{k-1}, \sigma_j^k])$, where

$$[\sigma_i^{k-1}, \sigma_j^k] = \begin{cases} +1 & +\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ -1 & -\sigma_i^{k-1} \in \partial_k \sigma_j^k \\ 0 & \sigma_i^{k-1} \notin \partial_k \sigma_j^k \end{cases}$$

Computation for Homology Basis

Cominatorial Laplace Operator

Construct linear operator $\Delta_k : C_k \to C_k$,

$$\Delta_k := \partial_k^T \partial_k + \partial_{k+1}\partial_{k+1}^T,$$

the eigen vectors of zero eigen values of Δ_k form the basis of $H_k(M, \mathbb{Z})$.

Smith Norm

The eigen vectors can be found using Smith norm of integer matrix. The computational cost is very high.

Simplicial Cohomology Group

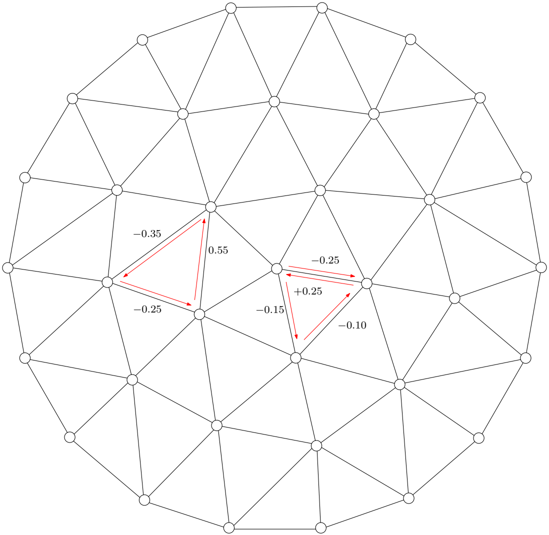


Figure: 1-Cochain.

Simplicial Cohomology Group

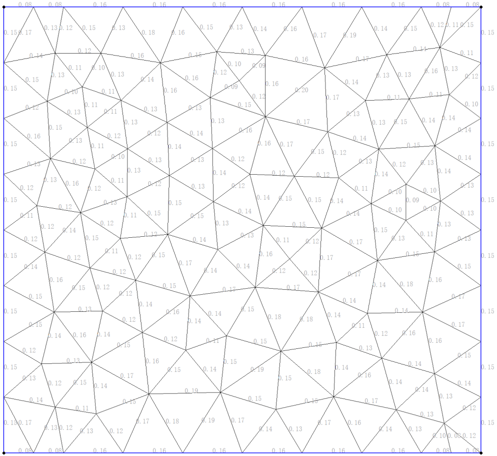


Figure: 1-Cochain.

Simplicial Cohomology Group

Definition (Cochain Space)

A k-cochain is a linear function

$$\omega : C_k \to \mathbb{Z}.$$

The k cochain space $C^k(\Sigma, \mathbb{Z})$ is a linear space formed by all the linear functionals defined on $C_k(\Sigma, \mathbb{Z})$. A k-cochain is also called a k-form.

Definition (Coboundary)

The coboundary operator $\delta_k : C^k(\Sigma, \mathbb{Z}) \to C^{k+1}(\Sigma, \mathbb{Z})$ is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \omega \in C^k(\Sigma, \mathbb{Z}).$$

Simplicial Cohomology Group

Example

M is a 2 dimensional simplicial complex, ω is a 1-form, then $\delta_1\omega$ is a 2-form, such that

$$
\begin{aligned}
\delta_1\omega([v_0, v_1, v_2]) &= \omega(\partial_2[v_0, v_1, v_2]) \\
&= \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0])
\end{aligned}
$$

Cohomology

Coboundary operator is similar to differential operator. δ_0 is the gradient operator, δ_1 is the curl operator.

Definition (closed forms)

A k-form is closed, if $\delta_k \omega = 0$.

Definition (Exact forms)

A k-form is exact, if there exists a $k-1$ form σ, such that

$$\omega = \delta_{k-1}\sigma$$

Cohomology

suppose $\omega \in C^k(\Sigma)$, $\sigma \in C_k(\Sigma)$, we denote the pair

$$\langle \omega, \sigma \rangle := \omega(\sigma).$$

Theorem (Stokes)

$$\langle d\omega, \sigma \rangle = \langle \omega, \partial\sigma \rangle.$$

Theorem

$$\delta^k \circ \delta^{k-1} \equiv 0.$$

All exact forms are closed. The curl of gradient is zero.

Cohomology

The difference between exact forms and closed forms indicates the topology of the manifold.

Definition (Cohomology Group)

The k-dimensional cohomology group of Σ is defined as

$$H^n(\Sigma, \mathbb{Z}) = \frac{ker\delta^n}{img\delta^{n-1}}.$$

Two 1-forms ω_1, ω_2 are cohomologous, if they differ by a gradient of a 0-form f,

$$\omega_1 - \omega_2 = \delta_0 f.$$

Homology vs. Cohomology

Duality

$H_1(\Sigma)$ and $H^1(\Sigma)$ are dual to each other. suppose ω is a closed 1-form, σ is a closed 1-chain, then the pair $\langle \omega, \sigma \rangle$ is a bilinear operator.

Definition (dual cohomology basis)

suppose a homology basis of $H_1(\Sigma)$ is $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, the dual cohomology basis is $\{\omega_1, \omega_2, \cdots, \omega_n\}$, if and only if

$$\langle \omega_i, \gamma_j \rangle = \delta_i^j.$$

Cohomology was introduced by H. Whitney in order to represent stiefel whitney class characteristic class. Prof. Chern learned it from Whitney.

Simplicial Mapping

Definition (simplicial mapping)

Suppose M and N are simplicial complexes, $f : M \to N$ is a continuous map, $\forall \sigma \in M$, σ is a simplex, $f(\sigma)$ is a simplex.

For each simplex, we can add its gravity center, and subdivide the simplex to multiple ones. The resulting complex is called the gravity center subdivision.

Theorem

Suppose M and N are simplicial complexes embedded in $\mathbb{R}^n$, $f : M \to N$ is a continuous mapping. Then for any $\epsilon > 0$, there exists gravity subdivisions $\tilde{M}$ and $\tilde{N}$, and a simplicial mapping $\tilde{f} : \tilde{M} \to \tilde{N}$, such that

$$\forall p \in |M|, |f(p) - \tilde{f}(p)| < \epsilon.$$

Simplicial Approximation

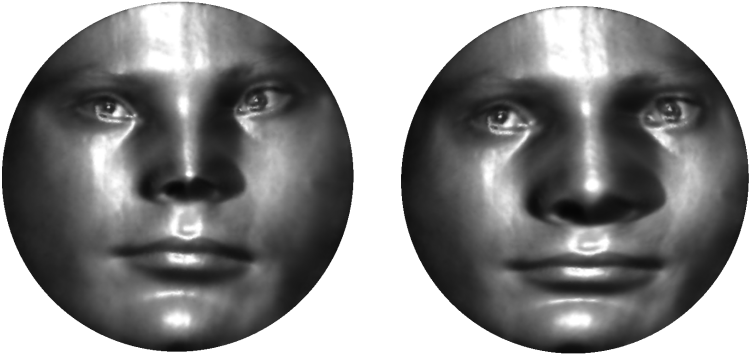


Figure: A planar map.

Simplicial Approximation

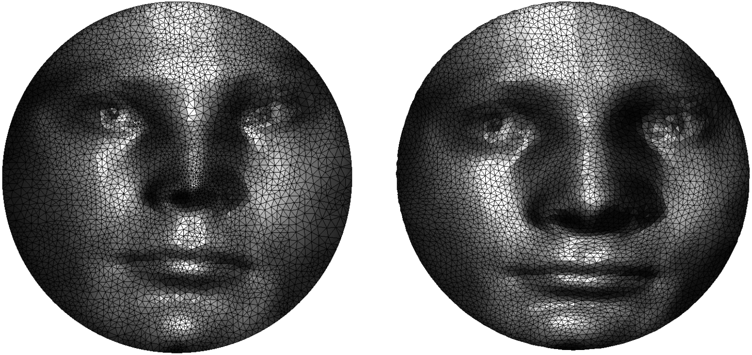


Figure: A planar map.

Simplicial Mapping

Definition (Pull-Back Map)

If $f : M \to N$ is a continuous map, then f induces a homomorphism $f_* : H_1(M) \to H_1(N)$, which push forward the chains of M to the chains in N. Similarly, f induces a pull back map $f^* : H^k(N) \to H^k(M)$. Suppose $\sigma \in C_1(M)$, $\omega \in C^1(N)$,

$$f^*\omega(\sigma) = \omega(f_*\sigma) = \omega(f(\sigma)).$$

Degree of a mapping

Suppose M and N are two closed surfaces. $H_2(M,\mathbb{Z}) = \mathbb{Z}$, $H_2(N,\mathbb{Z}) = \mathbb{Z}$, suppose $[M]$ is the generator of $H_2(M)$, which is the union of all faces. similarly, $[n]$ is the generator of $H_2(N)$. $f : M \to N$ is a continuous map. Then

$$f_* : \mathbb{Z} \to \mathbb{Z},$$

must has the form $f_*(z) = cz, c \in \mathbb{Z}$.

Definition (Mapping Degree)

$f_*([M]) = c[N]$, then the integer c is the degree of the map.

map degree is the algebraic number of pre-images $f^{-1}(q)$ for arbitrary point $q \in N$, which is independent of the choice the point q.

Degree of a mapping

Example (Gauss-Bonnet)

$G : S \to \mathbb{S}^2$ is the Gauss map, which maps the point p to its normal $\mathbf{n}(p)$, then $deg(G) = 1 - g$. The total area of the image is $4\pi\deg(G) = 2\pi\chi(S)$.

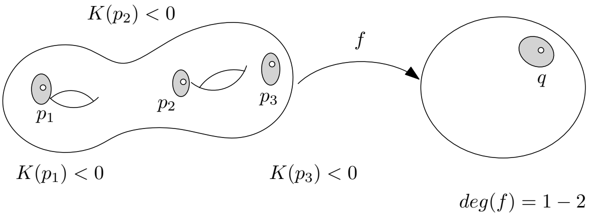


Figure: Map degree and Gauss-Bonnet theorem proof.

High dimensional Gauss-Bonnet theorem was first proved by Allendoerfer and Weil, Prof. Chern used different method to reprove it.

Algorithm for Cohomology Group

Algorithm for $H^1(M,\mathbb{R})$

Input: A genus g closed triangle mesh M;
Output: A set of basis of $H^1(M,\mathbb{R})$

1. Compute a set of basis of $H_1(M,\mathbb{Z})$, denoted as
$$\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\},$$
2. for each γ_i, slice M along *gamma*$_i$, to obtain a mesh with two boundaries M_i, $\partial M_i = \gamma_i^+ - \gamma_i^-$;
3. set a 0-form τ_i on M_i, such that $\tau_i(v) = 1$ for all $v \in \gamma_i^+$ and $\tau_i(w) = 0$, for all $w \in \gamma_i^-$; set $\omega_i = d\tau_i$;
4. All $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ form a basis of $H^1(M,\mathbb{R})$.