Existence of the Solution to Discrete Surface Ricci Flow

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Discrete Surface Curvature Flow Theorem

Vertex Scaling

Definition (Vertex Scaling)

Two triangulated PL surface $(S, V, \mathcal{T}, d)$ and $(S, V, \mathcal{T}, d')$ are said to differ by a vertex scaling, if $\exists \lambda : V(\mathcal{T}) \to \mathbb{R}_{>0}$, such that $d' = \lambda * d$ on $E(\mathcal{T})$, where

$$\lambda * d(u, v) = \lambda(u)\lambda(v)d(u, v).$$

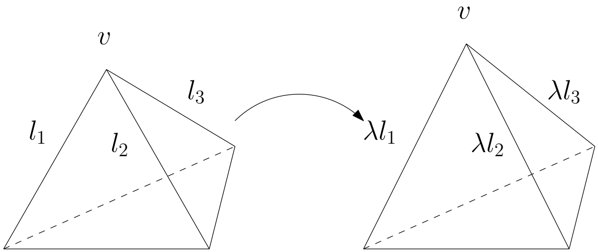


Figure: vertex scaling.

Discrete Conformal Equivalence

Definition (Gu-Luo-Sun-Wu)

Two PL metrics d, d' on a closed marked surface (S, V) are *discrete conformal*, if they are related by a sequence of two types of moves: vertex scaling and edge flip preserving Delaunay property.

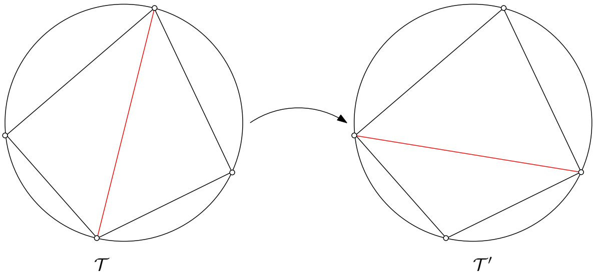


Figure: Edge flip, both triangulations are Delaunay.

Discrete Conformal Equivalence

Given a PL metric d on (S, V), produce a Delaunay triangulation $\mathcal{T}$ of (S, V),

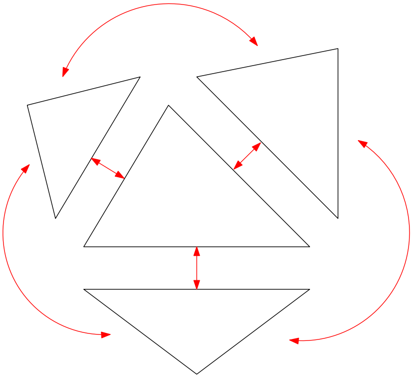
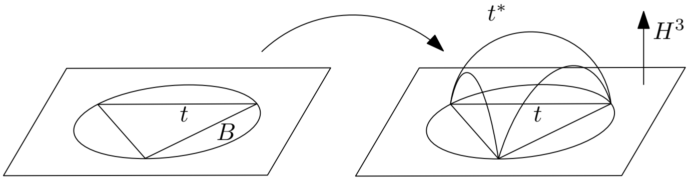


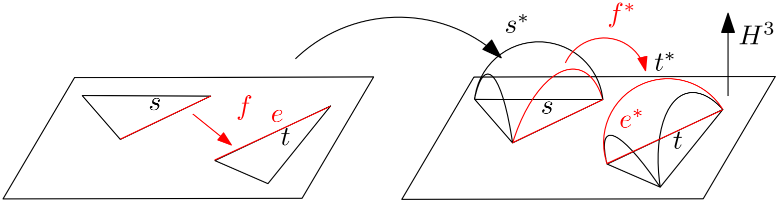
Figure: (S, V) with PL metric d, the triangulation is Delaunay.

Discrete Conformal Equivalence

Each face $t \in \mathcal{T}$ is associated an ideal hyperbolic triangle:

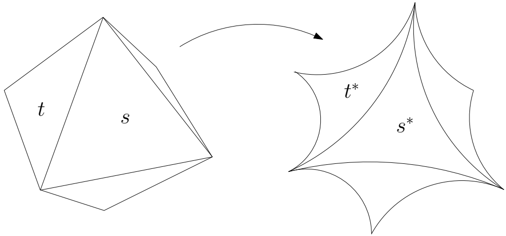


If $t, s \in \mathcal{T}$ glued by isometry f along e, then t^* and s^* are glued by the same f^* alonge e^*,



Discrete Conformal Equivalence

This induces a hyperbolic metric d^* on $S - V$.



Motivated by the important work of Bobenko-Pinkall-Springborn, equivalent to the previous defintion using vertex scaling and Delaunay condition.

Definition (Gu-Luo-Sun-Wu, JDG 2018)

Two PL metrics d_1 and d_2 on (S, V) are *discrete conformal* iff d_1^* and d_2^* are isometric by an isometry homotopic to identity on $S - V$.

Existence of the metric

Theorem (Gu-Luo-Sun-Wu)

Given a PL metric d on a closed marked surface (S, V), and curvature $K^* : V \to (-\infty, 2\pi)$, such that K satisfies the Gauss-Bonnet condition $\sum K(v) = 2\pi\chi(S)$, there there is a d^* discrete conformal to d, and d^* realizes the curvature K^*.

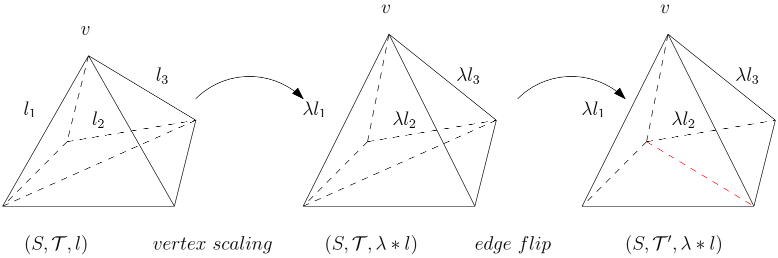


Figure: Discrete surface Yamabe flow.

Discrete Conformal Equivalence

Convex Optimization

Using Newton's method to minimize the following energy

$$\min_{\lambda} \int^{(\lambda_1, \lambda_2, \ldots, \lambda_n)} \sum_{v} (K^*(v) - K(v)) d \log \lambda(v),$$

such that $\Pi_v \lambda(v) = 1$. During the optimization, keep the triangulation always to be Delaunay.

Proof of the Discrete Surface Curvature Flow Theorem

Marked Surface

Definition (Marked Surface)

Let S be a closed topological surface, $V = \{v_1, v_2, \cdots, v_n\} \subset S$ is the set of distinct points, satsifying negative Euler number condition $\chi(S - V) < 0$. We call (S, V) a marked surface.

We consder the polyhedral metric $\mathbf{d}$ on the marked surface (S, V), with cone singularities at vertices.

Discrete Conformal Equivalence

Definition (Discrete Conformal Equivalence)

Two polyhedral metrics $\mathbf{d}$ and $\mathbf{d}'$ on a marked surface (S, V) are discrete conformal equivalent, if there is a series polyhedral metrics on (S, V),

$$\mathbf{d} = \mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m = \mathbf{d}'$$

and a series of triangulations $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m$, such that

1. every triangulation $\mathcal{T}_k$ is Delaunay on the metric $\mathbf{d}_k$;
2. if $\mathcal{T}_i = \mathcal{T}_{i+1}$, then there is a conformal factor $\mathbf{u} : V \to \mathbb{R}$, such that $\mathbf{d}_{i+1} = \mathbf{u} * \mathbf{d}_i$, namely the two polyhedral metrics differ by a vertex scaling operation;
3. if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometric transformation $h : (S, V, \mathbf{d}_i) \to (S, V, \mathbf{d}_{i+1})$, this transformation is homotopic to the identity map of (S, V), preserving the vertices.

Main Theorem

Existence and Uniqueness of the Solution to the Discrete Surface Ricci Flow:

Theorem (Gu-Luo-Sun)

Suppose $(S, V, \mathbf{d})$ *is a closed polyhedral surface, the for any* $K^* : V \to (-\infty, 2\pi)$, *satisfying the Gauss-Bonnet condition* $\sum_{v\in V} K^*(v) = 2\pi\chi(S)$, *there exists a polyhedral metric* $\mathbf{d}^*$

1. $\mathbf{d}^*$ *is discrete conformal equivalent to the metric* $\mathbf{d}$;
2. $\mathbf{d}^*$ *induces the discrete Gaussian curvature* K^*.

All such kind of polyhedral metrics differ by a global scaling. Furthermore, $\mathbf{d}^*$ *can be obtained by discrete surface Ricci flow.*

Uniformization

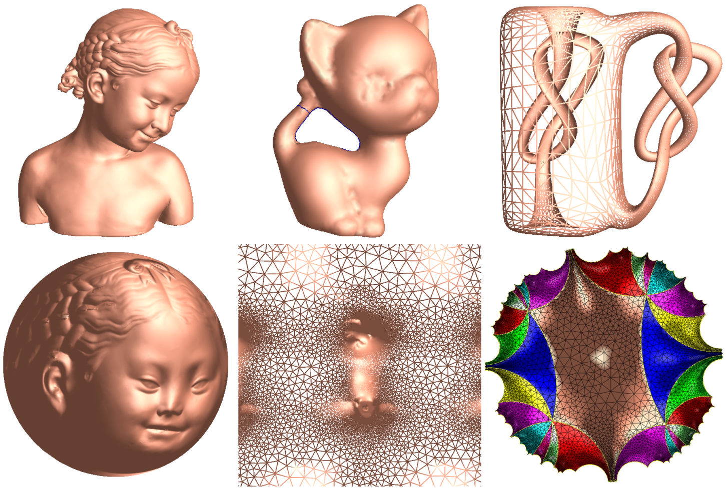


Figure: Closed surface uniformization.

Discrete Uniformization Theorem

Corollary (Gu-Luo-Sun)

Suppose $(S, V, \mathbf{d})$ *is a closed polyhedral surface, then there exists a polyhedral metric* $\mathbf{d}^*$, $\mathbf{d}^*$ *and the metric* $\mathbf{d}$ *are discrete conformal equivalent,* $\mathbf{d}^*$ *induces constant discrete Gaussian curvature* $2\pi\chi(S)/|V|$. *Such kind of polyhedral metrics differ by a global scaling.*

Teichmüller Space of Polyhedral Metrics

Definition (Equivalent Polyhedral Metrics)

Two polyhedral metrics $\mathbf{d}$ and $\mathbf{d}'$ on a marked surface (S, V) are equivalent, if there is an isometric tranformation $h : (S, V, \mathbf{d}) \to (S, V, \mathbf{d}')$, and h is homotopic to the identity map of (S, V), namely h preserves V.

Definition (Teichmüller Space of Polyhedral Metrics)

All the equivalence classes of polyhedral metrics on a marked surface (S, V) form the Teichmüller Space of polyhedral metrics.

$$T_{pl}(S, V) = \{\mathbf{d} | \text{polyhedral metrics on } (S, V)\} / \{\text{isometries} \sim \text{identity}\,(S, V)\}$$

Atlas of the Teichmüller Space of PL Metrics

Theorem (Troyanov)

Suppose (S, V) is a closed marked surface, the Teichmüller space of polyhedral metrics $T_{pl}(S, V)$ is homeomorphic to the Euclidean space $\mathbb{R}^{-3\chi(S-V)}$.

Definition (Local Chart of the Teichmüller Space of PL Metrics)

Suppose $\mathcal{T}$ is a triangulation of (S, V), its edge length function defines a polyhedral metric,

$$\Phi_{\mathcal{T}} : \mathbb{R}_{\triangle}^{E(\mathcal{T})} \to T_{pl}(S, V) \tag{1}$$

this gives a local chart of the Teichmüller space. Where the domain

$$\mathbb{R}_{\triangle}^{E(\mathcal{T})} = \left\{ x \in \mathbb{R}_{>0}^{E(\mathcal{T})} \middle| \text{for any } e_i, e_j, e_k \text{form a triangle }, x(e_i) + x(e_j) > x(e_k) \right\} \tag{2}$$

is a convex set, and is injective. We use $\mathcal{P}_{\mathcal{T}}$ to represent the image of $\Phi_{\mathcal{T}}$. Then $(\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})$ is a local chart of $T_{pl}(S, V)$.

Atlas of the Teichmüller Space of PL Metrics

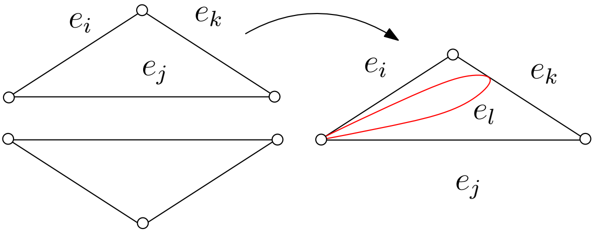


Figure: topological, not geometric triangulation.

If we edge swap e_k to e_l ot obtain the new triangulation $\mathcal{T}'$. Then under the metric $\mathbf{d}$, the topological triangle $\{e_j, e_l, e_j\}$ doesn't satisfy the triangle inequality. This shows the topological triangulation $\mathcal{T}'$ is not geometric.

$$\mathcal{P}(\mathcal{T}) \neq T_{pl}(S, V)$$

One chart can't cover the whole Teichmüller space $T_{pl}(S, V)$.

Definition (Atlas of Teichmüller Space of PL Metrics)

Suppose (S, V) is a closed marked surface, the atlas of $T_{pl}(S, V)$ consists of local coordinate charts $(\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})$, where $\mathcal{T}$ exhausts all possible triangulation.

$$\mathcal{A}(T_{pl}(S, V)) = \bigcup_{\mathcal{T}} (\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1}). \tag{3}$$

Lemma (Real Analytic Manifold)

Suppose (S, V) is a closed marked surface, then the Teichmüller space of polyhedral metrics $T_{pl}(S, V)$ is a real analytic manifold.

Teichmüller Space of Decorated Hyperbolic Metrics

Definition (Equivalent decorated hyperbolic metrics)

Two decorated hyperbolic metrics $(\mathbf{h}, \mathbf{w})$ and $(\mathbf{h}', \mathbf{w}')$ on a closed marked surface (S, V) are equivalent, if there is an isometric transformation

$$h : (S, V, \mathbf{h}, \mathbf{w}) \to (S, V, \mathbf{d}', \mathbf{w}'),$$

which is homotopic to the identity map of (S, V), and preserves the horospheres.

Definition (Teichmüller Space of Decorated Hyperbolic Metrics)

Given a closed marked surface (S, V), $\chi(S - V) < 0$, then all the decorated hyperbolic metric on it form the Teichmüller space:

$$T_D(S, V) = \frac{\{(\mathbf{h}, \mathbf{w}) | (S, V) \text{decorated hyperbolic metrics}\}}{\{\text{isometries} \sim \text{identity of } (S, V) \text{preserving horospheres}\}} \tag{4}$$

Teichmüller Space of Decorated Hyperbolic Metrics

Definition (Local Chart of the Teichmüller Space)

Suppose $\mathcal{T}$ is a triangulation of (S, V), the hyperbolic edge length function determines a decorated hyperbolic metric,

$$\Psi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to T_D(S, V) \tag{5}$$

which gives a local coordinate of the Teichmüller space. Let $\mathcal{Q}_{\mathcal{T}}$ be the image of $\Psi_{\mathcal{T}}$, then $(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1})$ form a local chart of $T_D(S, V)$.

Definition (Atlas of the Teichmüller Space)

Every triangulation of the marked closed surface (S, V) corresponds to a local chart $(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1})$. By exhausting all the possible triangulations, the union of all the local charts forms the atlas:

$$\mathcal{A}(T_D(S, V)) = \bigcup_{\mathcal{T}} \left(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1}\right).$$

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Equivalent Complete Hyperbolic Metrics)

Two complete hyperbolic metrics $\mathbf{h}$ and $\mathbf{h}'$ with finite area on a marked surface $(S-V)$ are equivalent, if there is an isometric transformation

$$h:(S-V,\mathbf{h})\to(S-V,\mathbf{h}'),$$

furthermore h is homotopic to the identity automorphism of $S-V$.

Definition (Teichmüller Space of Complete Hyperbolic Metrics)

All the complete hyperbolic metrics with finite area on a marked surface $S-V$, $\chi(S-V)<0$, form the Teichmüller space,

$$T_H(S-V)=\frac{\{\mathbf{h}|\text{complete hyperbolic metrics with finite area on }(S-V)\}}{\{\text{isometries}\sim\text{identity of }(S-V)\}} \tag{6}$$

Teichmüller Space of Complete Hyperbolic Metrics

Lemma (Local Coordinates)

Suppose $\mathbf{h}$ *is a complete hyperbolic metric on* $S - V$ *with finite area, the shear coordinate function is* $s : E(\mathcal{T}) \to \mathbb{R}$, *then for any* $v \in V$, *we have the relation*

$$\sum_{e \sim v} s(e) = 0. \tag{7}$$

Definition (Local Chart of the Teichmüller Space)

Let $\mathcal{T}$ be a triangulation of (S, V), its shear coordinates uniquely determines a complete hyperbolic metric with finite area,

$$\Theta_{\mathcal{T}} : \Omega_{\mathcal{T}} \to T_H(S - V) \tag{8}$$

this gives local coordinates of the Teichmüller space, where

$$\Omega_{\mathcal{T}} = \left\{ x \in \mathbb{R}^{E(\mathcal{T})} \middle| \sum_{e \sim v} x(e) = 0, \ \forall v \in V(\mathcal{T}) \right\}.$$

Then $(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1})$ form a local chart of $T_H(S - V)$.

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Atlas of the Teichmüller Space)

Let $\mathcal{T}$ be an arbitrary triangulation of (S, V), then $\mathcal{T}$ corresponds to a local chart $(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1})$. By exhausting all possible triangualtions of (S, V), all the local charts form an atlas of the Teichmüller space $T_H(S - V)$,

$$\mathcal{A}(T_H(S - V)) = \bigcup_{\mathcal{T}} \left(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1}\right).$$

Teichmüller Space of Complete Hyperbolic Metrics

Lemma

Given a closed marked surface (S, V), $\chi(S - V) < 0$

$$T_D(S, V) = T_H(S - V) \times \mathbb{R}_{>0}^{|V|}. \tag{9}$$

Proof.

Any decorated hyperbolic metric on $(S, V, \mathcal{T})$ can be represented as $(\mathbf{h}, \mathbf{w})$, where $\mathbf{h}$ is a complete hyperbolic metric on $S - V$ with finite area, $\mathbf{h} \in T_H(S - V)$; $\mathbf{w}$ is the lengths of intersections between the horospheres and the surface. □

Diffeomorphisms Among Teichmüller Spaces

The Teichmüller space of all PL metrics has a cell decomposition, each cell

$$D_{pl}(\mathcal{T}) = \{[\mathbf{d}] \in T_{pl}(S, V) | \mathcal{T} \text{ is Delaunay under } \mathbf{d}\}$$

We show $D_{pl}(\mathcal{T})$ is simply connected. We change the edge length $x(e)$ to Rivin coordinates $y(e)$, $y(e) = \alpha + \alpha'$. Then the edge lengths of $(S, V, \mathcal{T}, \mathbf{d})$ are determined by the Rivin's coordinates unique to a scaling,

$$D_{pl}(\mathcal{T}) = \{y(e) \in (0, \pi) | e \in E(\mathcal{T})\} \times \mathbb{R}_{>0}$$

is a convex set. D_{pl} is simply connected.

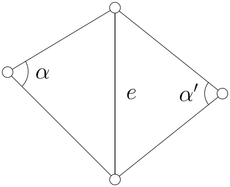


Figure: Rivin coordinates.

Diffeomorphisms Among Teichmüller Spaces

Cell Decomposition of $T_{pl}(S, V)$

The Teichmüller of the PL metrics has the cell decomposition:

$$T_{pl}(S, V) = \bigcup_{\mathcal{T}} D_{pl}(\mathcal{T}).$$

Cell Decomposition of $T_D(S, V)$

The Teichmüller space of the decorated hyperbolic metrics has the cell decomposition:

$$T_D(S, V) = \bigcup_{\mathcal{T}} D(\mathcal{T}).$$

where the cell

$$D(\mathcal{T}) = \{(\mathbf{d}, \mathbf{w}) \in T_D(S, V) | \mathcal{T} \text{ is Delaunay under } (\mathbf{d}, \mathbf{w})\}.$$

Diffeomorphisms Among Teichmüller Spaces

$$\begin{tikzcd}
D_{pl}(\mathcal{T}) \arrow[r, "\Psi_{\mathcal{T}} \circ \Phi_{\mathcal{T}}^{-1}"] & D(\mathcal{T}) \\
& \mathbb{R}_{\Delta}^{E(\mathcal{T})} \arrow[ul, "\emptyset_{\mathcal{T}}"'] \arrow[u, "\Psi_{\mathcal{T}}"']
\end{tikzcd}$$

We use Penner's λ-length to establish the diffeomorphism between two cells,

$$A_{\mathcal{T}} = \Psi_{\mathcal{T}} \circ \Phi_{\mathcal{T}}^{-1} : D_{pl}(\mathcal{T}) \to D(\mathcal{T}), \quad x(e) \mapsto 2 \ln x(e)$$

Penner's λ-length maps Euclidean Delaunay triangulation to decorated hyperbolic Delaunay triangulation. Furthermore Delaunay property implies triangle inequality, hence $A_{\mathcal{T}}$ is a diffeomorphism.

Diffeomorphisms Among Teichmüller Spaces

Suppose triangulations $\mathcal{T}$ and $\mathcal{T}'$ differ by an edge swap, consider a polyhedral metric $[d] \in D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')$, then under d, there are four co-circle vertices in (T) and $(T)'$. By Ptolemy equality, we obtain for any $x \in \Phi_{\mathcal{T}}^{-1}(D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}'))$,

$$\Phi_{\mathcal{T}}^{-1} \circ \Phi_{\mathcal{T}'}(x) = \Psi_{\mathcal{T}}^{-1} \circ \Psi_{\mathcal{T}'}(x)$$

this is equivalent to

$$A_{\mathcal{T}}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')} = A_{\mathcal{T}'}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')}$$

In this way, we glue the piecewise diffeomorphisms $A_{\mathcal{T}}$ to form a global diffeomorphism:

$$A : T_{pl}(S,V) \to T_D(S,V), \quad A|_{D_{pl}(\mathcal{T})} = A_{\mathcal{T}}|_{D_{pl}(\mathcal{T})}$$

Further proof shows this mapping is globally C^1 diffeomorphic.

Existence Proof

First, we construct a map: $F : \Omega_u \to \Omega_K$,

$$\Omega_u \xrightarrow{\exp} \{p\} \times \mathbb{R}^{|V|}_{>0} \to T_D(S,V) \xrightarrow{A^{-1}} T_{pl}(S,V) \xrightarrow{K} \Omega_K \tag{10}$$

where the domain Ω_u is the intersection between the discrete conformal factor space and the Euclidean hyperplane

$$\Omega_u = \mathbb{R}^n \cap \left\{ \mathbf{u} \middle| \sum_{i=1}^{n} u_i = 0 \right\} \tag{11}$$

the range Ω_K is the discrete curvature space,

$$\Omega_K = \left\{ \mathbf{K} \in (-\infty, 2\pi)^n \middle| \sum_{i=1}^{n} K_i = 2\pi\chi(S) \right\} \tag{12}$$

both of them are open sets in the Euclidean space $\mathbb{R}^{n-1}$. Because $A : T_{pl}(S,V) \to T_D(S,V)$ is C^1, $K : T_{pl}(S,V) \to \mathbb{R}^n$ is real analytic, hence F is C^1.

Existence Proof

We show that the map $F : \Omega_u \to \Omega_K$ is injective. Consider the convexity of the entropy energy

$$\mathcal{E}(\mathbf{u}) = \int^{\mathbf{u}} \sum_{i=1}^{n} K_i du_i.$$

The Hessian Matrix is the discrete Laplace-Beltrami operator, hence the entropy is strictly convex on the domain Ω_u. Furthermore, the domain Ω_u is convex, the gradient of the entropy is the current discrete curvature. Hence, the map $\mathbf{u} \mapsto \nabla\mathcal{E}(\mathbf{u}) = \mathbf{K}(\mathbf{u})$ is injective.

Existence Proof

We then show that the map $F : \Omega_u \to \Omega_K$ is surjective. This requires domain inviarance theorem.

Theorem (Invariance of Domain)

Suppose U is a domain (connected open set) in $\mathbb{R}^n$, if $f : U \to \mathbb{R}^n$ is continuous and injective, then $V = f(U)$ is open, and f is a homeomorphism between U and V.

Because both Ω_u and Ω_K are all $n-1$ dimensional open sets, F is continuous and injective, hence $F(\Omega_u)$ is an open set. And $F : \Omega_u \to F(\Omega_u)$ is homeomorphic. We need to show $\Omega_K = F(\Omega_u)$.

Existence Proof

Since $F(\Omega_u)$ is open, we need to show $F(\Omega_u)$ is closed in Ω_K. We take a sequence $\{x_k\} \subset \Omega_u$, such that x_k leaves all the compact sets in Ω_u. We need to show $F(x_k)$ leaves all the compact sets in Ω_K. We need the Akiyoshi theorem:

Theorem (Akiyoshi(2001))

For any complete hyperbolic metric d on $S - V$ with finite area, there exists finite number of isotopy classes of triangulations $\mathcal{T}$, such that

$$[d] \times \mathbb{R}^n_{>0} \bigcap D(\mathcal{T}) \neq \emptyset.$$

Furthermore, there is finite number of triangulations $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$, such that for any decoration $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Delaunay triangulation of (d, w) is isotopic to one of such $\mathcal{T}_i$.

By Akiyoshi theorem, $\{p\} \times \mathbb{R}^n_{>0}$ intersects $T_D(S, V)$ at a finite number of cells, hence we can assume the Delaunay triangulation $\mathcal{T}$ is fixed.

Existence Proof

$\{x_k\}$ leaves all the compact sets in Ω_u. By taking subsequences, we may assume that for each vertex v_i, $\lim_k x_i^{(k)} = t_i$ exists in $[-\infty, +\infty]$. Due to the normalization that $\sum_i x_i^{(k)} = 0$ and $x^{(k)}$ doesn't converge to any vector in Ω_u, there exists $t_i = \infty$ and $t_j = -\infty$. We label vertices by black and white. The vertex v_i is black if and only if $t_i = -\infty$ and white otherwise.

Lemma (Coloring)

1. *There doesn't exist a triangle $\tau \in \mathcal{T}$ with exactly two white vertices.*
2. *If $\Delta v_1 v_2 v_3$ is a triangle in $\mathcal{T}$ with exactly one white vertex at v_1, then the inner angle at v_1 converges to 0 as $k \to \infty$ in the metric d_k.*

Existence Proof

Proof.

To see (1), suppose otherwise, there exists a Euclidean triangle of lengths $a_i e^{u_j^{(n)}+u_k^{(n)}}$, $\{i,j,k\}=\{1,2,3\}$, where $\lim_n u_i^{(n)} > -\infty$ for $i=2,3$ and $\lim_n u_1^{(n)} = -\infty$. By the triangle inequality, we have

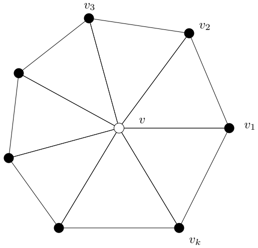
$$a_2 e^{u_1^{(n)}+u_3^{(n)}} + a_3 e^{u_1^{(n)}+u_2^{(n)}} > a_1^{u_2^{(n)}+u_3^{(n)}}.$$

This is the same as

$$a_2 e^{-u_2^{(n)}} + a_3 e^{-u_3^{(n)}} > a_1^{-u_1^{(n)}}.$$

However, the left-hand-side is bounded, the right-hand-side tends to ∞. The contradiction shows (1) holds.
To see (2), the triangle is similar to one with edge lengths, $\{a_1 e^{-u_1^{(n)}}, a_2 e^{-u_2^{(n)}}, a_3 e^{-u_3^{(n)}}\}$, converge to $\{c,\infty,\infty\}$, hence the angle α_1 tends to 0.

Existence Proof



We now finish the proof of $F(\Omega_u) = \Omega_k$ as follows. Since the surface S is connected, there exists an edge e whose end points v, v_1 have different colors. Assume v is white and v_1 is black. Let $v_1, \ldots, v_k$ be the set of all vertices adjacent to v so that v, v_i, v_{i+1} form vertices of a triangle and let $v_{k+1} = v_1$. Now apply above lemma to triangle $\Delta v v_1 v_2$ with v white and $v1$ black, we conclude that v_2 must be black. Inductively, we conclude that all v_i's, for $i = 1, 2, \ldots, k$, are black. By part (2) of the above lemma, we conclude that the curvature of d_n at v tends to 2π. This shows that $F(\Omega_u^{(n)})$ tends to ∞ of Ω_k. Therefore $F(\Omega_u) = \Omega_k$. $\square$

Convergence Theorem

Main Theorem

Definition (δ-Triangulation)

Given a compact polyhedral surface $(S, V, \mathbf{d})$, a triangulation $\mathcal{T}$ is a δ-triangulation, $\delta > 0$, if all the inner angles of each face is in the interval $(\delta, \pi/2 - \delta)$.

Definition ((δ, c) Subdivision Sequence)

Given a compact triangulated polyhedral surface $(S, \mathcal{T}, l^*)$, a geometric subdivision sequence $(\mathcal{T}_n, l_n^*)$ of $(\mathcal{T}, l^*)$ is a (δ, c) subdivision sequence, where $\delta > 0$, $c > 1$ are positive numbers, if every $(\mathcal{T}_n, l_n^*)$ is a δ-triangulation, and the edge lengths satisfy

$$l_n^*(e) \in \frac{1}{n}\left(\frac{1}{c}, c\right), \quad \forall e \in E(\mathcal{T}_n).$$

Convergence Theorem

In the above definition, polyhedral surfaces can be replaced by general Riemann surfaces, triangulations can be replaced by geodesic triangulations, the lengths are replaced by geodesic lengths, to obtain the so-called (δ, c) geodesic subdvision sequence.

Convergence Theorem

Theorem (Convergence of Discrete Curvature Flow)

Given a curved triangle with a Riemannian metric $(S, \mathbf{g})$, *three corner angles are* $\pi/3$. *Given a* (δ, c) *geodesic subdivision sequence* $(\mathcal{T}_n, L_n)$, *for any edge* $e \in E(\mathcal{T}_n)$, $L_n(e)$ *is geodesic length under the metric* $\mathbf{g}$. *Then there exists discrete conformal factor* $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$, *for n big enough,* $C_n = (S, \mathcal{T}_n, w_n * L_n)$, *such that*

a. C_n *is isometric to the planar equilateral triangle* $\triangle$, *and* C_n *is* $\delta_\Delta/2$*-triangulation, where the constant* δ_Δ *doesn't depend on the surface;*

b. *Discrete uniformization maps* $\varphi_n : C_n \to \triangle$, *satisfy*

$$\lim_{n\to\infty} \| \varphi_n|_{V(\mathcal{T}_n)} - \varphi|_{V(\mathcal{T}_n)} \|_\infty = 0, \tag{13}$$

uniformly converge to the smooth uniformization map $\varphi : (S, \mathbf{g}) \to (\triangle, dzd\bar{z})$.

Key Lemmas

Lemma

Suppose $(S, \mathbf{g}_1)$ *is a* C^2 *smooth compact surface, its boundary* ∂S *may be non-empty with corners,* $\mathbf{g}_2 = e^{2\mu}\mathbf{g}_1$ *is another Riemannian metric, conformal equivalent to the original metric, where the conformal factor* $\mu \in C^2(S)$ *is a* C^2 *smooth function. Then there exists constant* $c = c(S, \mathbf{g}_1, \mu)$*, such that for any geodesic connecting a pair of points* p *and* q*, or* γ *is a boundary curve segment,* $\gamma \subset \partial S$*, we have the estimate*

$$\left| l_{\mathbf{g}_2}(\gamma) - e^{\frac{\mu(p)+\mu(q)}{2}} l_{\mathbf{g}_1}(\gamma) \right| \leq c(S, \mathbf{g}_1, \mu) l^3_{\mathbf{g}_1}(\gamma)$$

Estimate

Theorem

Given a compact triangulated polyhedral surface $(S, \mathcal{T}, l^*)$, $(S, \mathcal{T}_n, l_n^*)$ *is a* (δ, c) *geometric subdivision sequence;* $(S, \mathcal{T}_n, l_n)$ *is another sequence of polyhedral metrics, satisfy the inequalities:* $|l_n(e) - l_n^*(e)| \leq c_0/n^3$, $\forall e \in E(\mathcal{T}_n)$, *where* $c_0 > 0$ *is a positive constant, then there exists a constant* $c_1 = c_1(l^*, \delta, c, c_0)$, *and discrete conformal factor* $v_n \in \mathbb{R}^{V(\mathcal{T}_n)}$, *for n big enough,*

1. $(\mathcal{T}_n, v_n * l_n)$ *is* $\delta/2$*-triangulation,*
2. $K_{v_n * l_n} = K_{l_n^*}$
3. *discrete conformal factor*
$$\|v_n\|_\infty \leq \frac{c_1(l^*, \delta, c, c_0)}{\sqrt{n}}$$

and we have the estimate

$$|l_n^*(e) - v_n * l_n(e)| \leq \frac{c_2(l^*, \delta, c, c_0)}{n\sqrt{n}}, \quad \forall e \in E(\mathcal{T}_n)$$

Planar Equilateral Triangle Subdivision Sequence

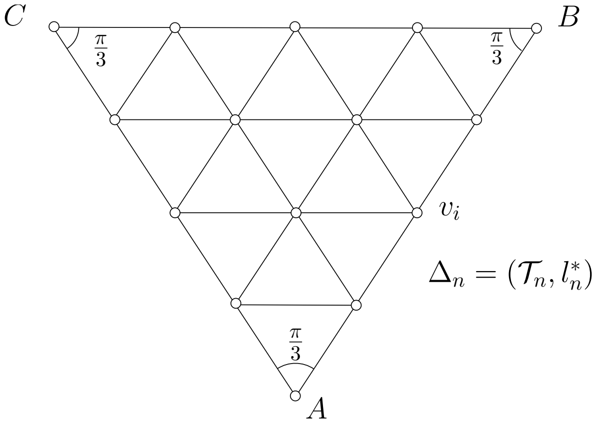


Figure: Planar equivaleteral triangle.

Planar Equilateral Triangle Subdivision Sequence

Planar equilateral triangle $\triangle ABC$, edge lengths are 1, corner angles are $\pi/3$. Every subdivision inserts middle points into the edges. After the n-th subdivision, the discrete surface is $\triangle_n$, its triangulation is $\mathcal{T}_n$, the PL metric is induced by the Euclidean planar metric $dzd\bar{z}$, represented as the length functions l_n^*. We use $\triangle_n = (\triangle, \mathcal{T}_n, l_n^*)$ to represent this discrete surface, obviously $\triangle_n$ is a (δ, c) subdivision sequence, where $(\delta_\triangle, c_\triangle) = (\pi/6 - \varepsilon, 1 - \varepsilon)$, $\varepsilon > 0$ is a arbitrarily small positive number.

Surface Geodesics Subdivision Sequence

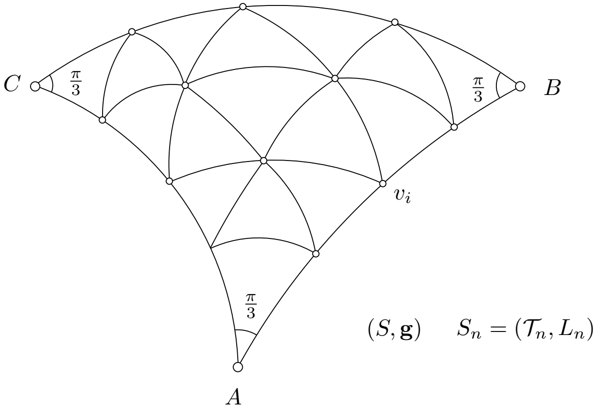


Figure: Smooth surface.

Given a C^2 smooth surface $(S, \mathbf{g})$, with three corner angles A, B, C. ∂S consists of three smooth curves, at each corner point, the intersection angle is $\pi/3$. There is a Riemann mapping $\varphi : (S, \mathbf{g}) \to \Delta$, which maps corners to corners, boundary curves to boundary line segments. The conformal factor induced by φ is a smooth bounded function, $\mu : S \to \mathbb{R}$,

$$\mathbf{g} =^{-4\mu} dz d\bar{z}.$$

Simultaniously, φ pulls back the triangulation $\mathcal{T}_n$ from Δ_n to S. We replace every edge on $\varphi^{-1}(\Delta_n)$ by geodesic segments, to obtain a geodesic triangulation, denoted as $S_n = (S, \mathbf{g}, \mathcal{T}_n, L_n)$, where L_n is the geodesic length of the triangulation $\mathcal{T}_n$. For any $\varepsilon > 0$, there exists $N(\varepsilon)$, when $n > N(\varepsilon)$, S_n is a (δ, c) geodesic subdivision sequence, $(\delta, c) = (\pi/6 - \varepsilon, 1 - \varepsilon)$.

Discretization Sequence

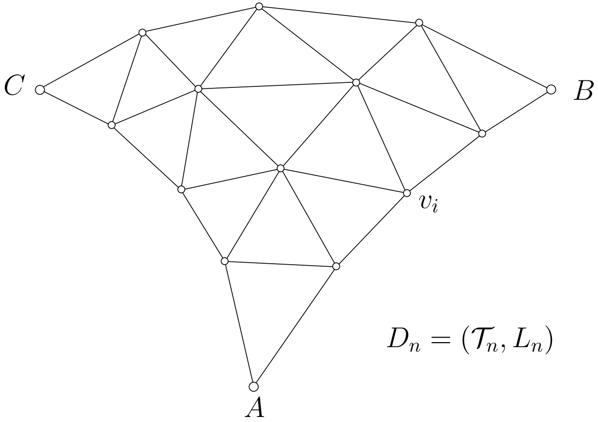
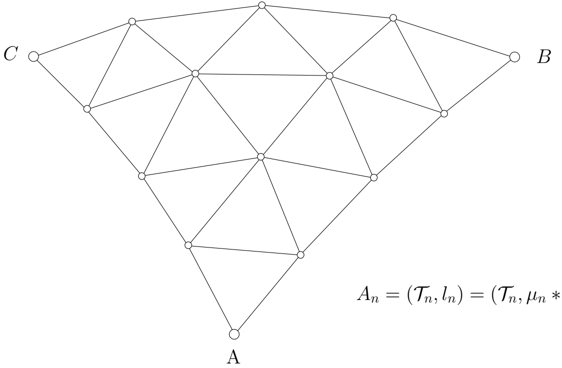


Figure: Discretization.

Discretization

We convert smooth geodesic subdivision sequence $S_n(\mathcal{T}_n, L_n)$ to PL surface $D_n = (\mathcal{T}_n, L_n)$. For any face $t \in \mathcal{T}_n$, with edges $\{e_i, e_j, e_k\}$, we use $\{L_n(e_i), L_n(e_j), L_n(e_k)\}$ as edge lengths to construct a Euclidean triangle, then isometrically glue these Euclidean triangles. Then (D_n, L_n) is a (δ, c)-subdivision sequence, where $c = c(S, \mathbf{g}, \mu)$.

Approximation Sequence



$$A_n = (\mathcal{T}_n, l_n) = (\mathcal{T}_n, \mu_n * L_n)$$

Figure: Approximation sequence.

Approximation Sequence

Smooth Riemann mapping $\varphi : (S, \mathbf{g}) \to \Delta$ induces conformal factor $\mu : S \to \mathbb{R}_{>0}$, $dzd\bar{z} = e^{4\mu}\mathbf{g}$. We define the discrete conformal factor: $\mu_n : V(\mathcal{T}_n) \to \mathbb{R}_{>0}$, for every vertex $v_i \in \mathcal{T}_n$,

$$\mu_n(v_i) = \mu(\varphi^{-1}(v_i)), \quad v_i \in \Delta_n, \quad \varphi^{-1} : \Delta_n \to S.$$

We use $D_n = (\mathcal{T}_n, L_n)$ to approximate (S_n, L_n), μ_n to approximate μ, then

$$A_n = (\mathcal{T}_n, \mu * L_n)$$

to approximate $\Delta_n = (\mathcal{T}_n, l_n^*)$. By the key lemma, for any $e \in \mathcal{T}_n$,

$$|l_n^*(e) - \mu_n * L_n(e)| \leq \frac{c_1}{n^3}, \quad c_1 = c_1(\mathbf{g}, \delta_S, c_S, dzd\bar{z}). \tag{14}$$

Compensation Sequence

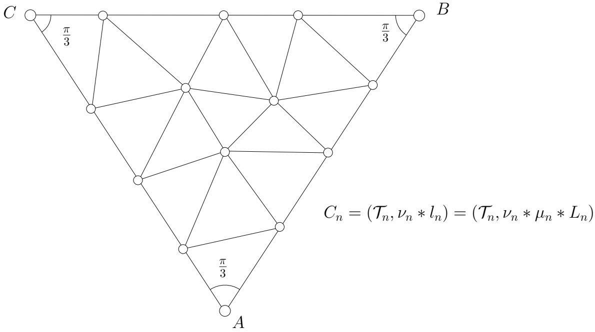


Figure: Compensation sequence.

Compensation Sequence

By the theorem, consider Δ_n and A_n sequences, there exists discrete conformal factor $\nu : V(\mathcal{T}_n) \to \mathbb{R}_{>0}$, such that

1. $C_n = (\mathcal{T}_n, \nu_n * (\mu * L_n))$ is a $\delta_\Delta/2$ triangulation;
2. $K_{\nu_n * \mu_n * L_n} = K_{l_n^*}$, this implies $C_n = \Delta$ is a planar equilateral triangle;
3. the L^∞ norm of the conformal factor

$$\|\nu_n\|_\infty \leq \frac{c_2(\mathbf{g}^*, \delta_S, c_1, c_S)}{\sqrt{n}}$$

4. for all edges $e \in E(\mathcal{T}_n,$

$$|l_n^*(e) - \nu_n * \mu_n * L_n(e)| \leq \frac{c_3(\mathbf{g}^*, \delta_S, c_1, c_S)}{n\sqrt{n}}$$

Outline of the Proof

The outline of the proof is as follows:

$$
\begin{tikzcd}
(S_n, L_n) \arrow[r, "\alpha_n"] \arrow[d, "\varphi"] & (D_n, L_n) \arrow[r, "\beta_n"] & (A_n, \mu_n * L_n) \arrow[d, "\gamma_n"'] \\
(\Delta_n, l_n^*) & & (C_n, \nu_n * \mu_n * L_n) \arrow[ll, "\varphi_n"']
\end{tikzcd}
$$

α_n: discretize the smooth surface using geodesic distance L_n; β_n: use smooth conformal factor μ_n to approximate uniformization map, $\mu_n * L_n$ and planar Euclidean length l_n^* differ by $O(n^{-3})$; γ_n: compensate the discrete error to obtain the discrete conformal factor ν_n, $\nu_n * \mu_n * L_n$ and l_n^* differ by $O(n^{-3/2})$; φ_n: piecewise linear map, the norm of the Beltrami coefficient of the quasi-conformal map φ_n is less than $C/\sqrt{n}$.

Proof for Convergence Theorem

Proof.

We construct a piece-wise linear map $\varphi_n : C_n \to \Delta_n$. Since C_n and Δ_n are equilateral triangles, by reflection, we can extend φ to $\tilde{\varphi}_n : \mathbb{C} \to \mathbb{C}$. Since C_n is a $\delta_\Delta/2$ triangulation, there exists a positive number $K > 1$, φ_n is a K-quasi-conformal map. We obtain a family of K quasi-conformal maps from the complex plane to itself $\{\tilde{\varphi}_n\}$. By the compactness of quasi-conformal maps, there exits a convergent subsequence $\{\tilde{\varphi}_{n_k}\}$, $\lim_{k\to\infty} \tilde{\varphi}_{n_k} = \tilde{\varphi}$.
Let $w_n = \mu_n + \nu_n$, by inequality 14, we obtain

$$\lim_{k\to\infty} \frac{l_n^*(e)}{w_n * L_n(e)} = 1.$$

Hence the dilatation of the limit map $\tilde{\varphi}$ $K = 1$. Hence $\tilde{\varphi}$ is conformal. □