Surface Immersion Regular Homotopy

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Differential Topology: Regular Homotopy Theory

Regular Homotopy

Definition (Parameterized Regular Closed Curve)

A parameterized regular closed curve is a path $\gamma : [0,1] \to \mathbb{R}^2$ satisfying the following conditions:

- $\gamma(0) = \gamma(1)$;
- γ has a well-defined, continuous derivative $\gamma' : [0,1] \to \mathbb{R}^2$;
- $\gamma'(0) = \gamma'(1)$;
- $\gamma'(t) \neq (0,0)$ for all $t \in [0,1]$.

A differentiable function $\gamma : [0,1] \to \mathbb{R}^2$ is a parameterized regular closed curve if and only if both γ and its derivative γ' are loops and γ' avoids the origin.

Regular Homotopy

Definition (Equivalent)

Two regular closed curves γ and δ are equivalent, denoted as $\gamma \sim \delta$, if they differ only by reparameterization, that is, if there is a continuous function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(t+1) = \eta(t) + 1$ and $\gamma(t) = \delta(\eta(t) \mod 1)$ for all t. If such a function η exists, its derivative must be positive everywhere.

It is easy to check that $\sim$ is an equivalence relation; the equivalence classes are called *regular closed curves*, and its elements are called *parameterizations*.

Regular Homotopy

Definition (Regular Homotopy)

A regular homotopy is a homotopy through parameterized regular closed curves, that is, a function $h : [0,1] \times [0,1] \to \mathbb{R}^2$ such that for all s, the function $t \mapsto h(s,t)$ is a parameterized regular closed curve.

Definition (Regular Homotopic Equivalence)

Two parameterized regular closed curves γ and δ are regular homotopic, denoted as $\gamma \sim_r \delta$ if there is a regular homotopy h such that $h(0,t) = \gamma(t)$ and $h(1,t) = \delta(t)$ for all t.

Regular Homotopy

Definition (Turning Number)

The turning number of a regular closed curve γ is the winding number of its derivative γ' around the origin (degree of the Gauss map).

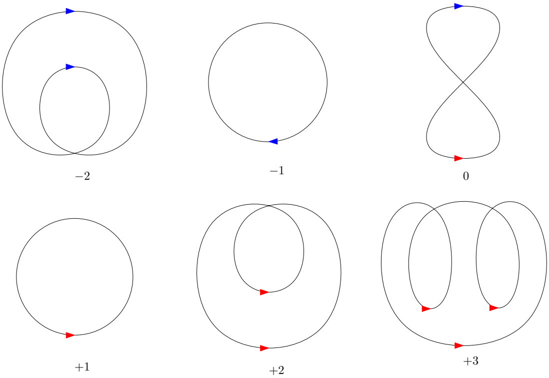


Figure: Turning number of regular curves.

Regular Homotopy

Theorem (Whitney-Graustein)

Two regular curves in $\mathbb{R}^2$ are regularly homotopic if and only if their turning numbers are equal.

Proof.

Let $h' : [0,1]^2 \to \mathbb{S}^1$ be a homotopy from γ' to δ'. A loop $\alpha : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ is the derivative of a regular closed curve if and only if its center of mass is the origin. Let $c : [0,1] \to \mathbb{R}^2$ be defined as $c(s) = \int_0^1 h'(s,t)dt$. Thus the function $h^*(s,t) = h'(s,t) - c(s)$ is a homotopy from γ' to δ'. $h(s,t) := \int_0^t h^*(s,u)du$ is a regular homotopy from γ to δ. □

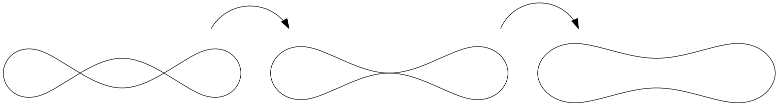
Regular Homotopy on the Sphere

Theorem

Suppose γ and δ are two regular curves on the sphere, γ is regular homotopic to δ if and only both of them have even or odd algebraic intersection numbers.

Proof.

Lift γ to the unit tangent bundle as $\tilde{\gamma}$ by $\gamma(t) \mapsto (\gamma(t), \gamma'(t)/|\gamma'(t)|)$, $\gamma \sim_r \delta$ if and only if $\tilde{\gamma}$ is homotopic to $\tilde{\delta}$. The fundamental group of the unit tangent bundle of $\mathbb{S}^2$ is $\mathbb{Z}_2$, there are only two regular homotopy classes. □



Fundamental Group $SO(3)$

Lemma

The fundamental domain of $SO(3)$ is $\mathbb{Z}_2$.

Proof.

Any orientation preserving rotation in $\mathbb{R}^3$ can be represented as (v, θ) where v is the rotation axis, $v \in \mathbb{S}^2$, and $\theta \in \mathbb{S}^1$. Therefore, $SO(3)$ is homeomorphic to $\mathrm{UTM}(\mathbb{S}^2)$, $\pi_1(SO(3)) = \mathbb{Z}_2$. □

Quaternion

Definition (Quaternion)

The space of quaternions $\mathbb{H}$ is a 4-dimensional vector space over $\mathbb{R}$ spanned by $1, i, j, k$,

$$\mathbb{H} = \{\rho + xi + yj + zk | (\rho, x, y, z) \in \mathbb{R}^4\}$$

with multiplicative relations:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

$$\mathrm{Re}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(1) \quad \mathrm{Im}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(i, j, k) \cong \mathbb{R}^3$$

Quaternion

The product of two quaternions: $q_\alpha = a_\alpha + b_\alpha i + c_\alpha j + d_\alpha k$, the product $q_1 q_2$ equals to

$$\begin{aligned} & a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ +& (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i \\ +& (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) j \\ +& (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) k \end{aligned}$$

$q_1 = (\rho_1, v_1)$, $q_2 = (\rho_2, v_2)$, where $\rho_\alpha \in \mathbb{R}$, $v_\alpha \in \mathbb{R}^3$,

$$q_1 q_2 = (\rho_1 \rho_2 - \langle v_1, v_2 \rangle_{\mathbb{R}^3}, \rho_1 v_2 + \rho_2 v_1 + v_1 \times_{\mathbb{R}^3} v_2).$$

Quaternion

Definition (Conjugate)

A conjugate is a linear operator on $\mathbb{H}$ such that for any $q = a + bi + cj + dk \in \mathbb{H}$ where $a, b, c, d \in \mathbb{R}$, the conjugation of q is

$$\bar{q} = a - bi - cj - dk.$$

The norm $|\cdot|$ is defined as

$$|\lambda|^2 = q\bar{q}.$$

The conjugate satisfies

$$\overline{\lambda\mu} = \bar{\mu}\bar{\lambda}$$

and

$$\text{Re}(\lambda\mu) = \text{Re}(\mu\lambda)$$

Quaternion and Rotation

Definition

An inner product $\langle , \rangle$ on $\mathbb{H}$ is defined such that for any two quaternions λ and $\mu \in \mathbb{H}$,

$$\langle \lambda, \mu \rangle_{\mathbb{H}} = \text{Re}(\lambda \bar{\mu})$$

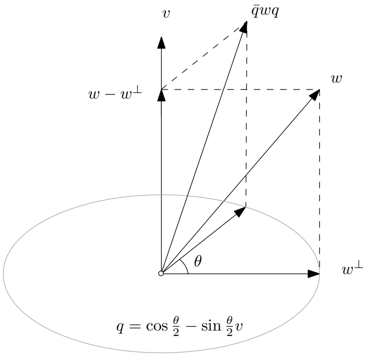
and the norm $|\cdot|$ is defined by

$$|\lambda|^2 = \langle \lambda, \lambda \rangle_{\mathbb{H}}.$$

Quaternion and Rotation

A rotation can be represented as a quaternion. The rotation axis is $v \in \mathbb{S}^2 \subset \text{Img}(\mathbb{H})$, $v = (x, y, z)$, the rotation angle is $\theta \in [0, 2\pi)$, the quaternion is

$$q = \cos\frac{\theta}{2} - \sin\frac{\theta}{2} v,$$



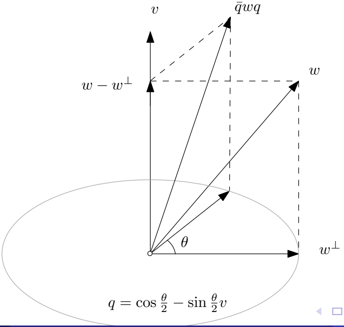
Quaternion and Rotation

Lemma

Let $\lambda \in \mathbb{H}$ be arbitrary, $\lambda = |\lambda|(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}v)$ form some $\theta \in \mathbb{R}$ and a unit vector $v \in \mathbb{S}^2 \subset Im(\mathbb{H})$. Then for any $w \in Im(\mathbb{H})$,

$$\bar{\lambda} w \lambda = |\lambda|^2((w - w^\perp) + \cos\theta w^\perp + \sin\theta v \times w^\perp)$$

where $w^\perp$ is the component of w perpendicular to v.



Let $\alpha = \theta/2$, $w = w^v + w^\perp$, then

$$
\begin{aligned}
&|\lambda|^2(\cos\alpha, \sin\alpha v)(0, w)(\cos\alpha, -\sin\alpha v)\\
=&|\lambda|^2(0 - \sin\alpha v \cdot w, \cos\alpha w + \sin\alpha v \times w)(\cos\alpha, -\sin\alpha v)\\
=&|\lambda|^2(-\sin\alpha\cos\alpha v \cdot w + (\cos\alpha w + \sin\alpha v \times w) \cdot \sin\alpha v,\\
&\sin^2\alpha v \cdot w v + \cos\alpha(\cos\alpha w + \sin\alpha v \times w)\\
&-(\cos\alpha w + \sin\alpha v \times w) \times \sin\alpha v)\\
=&|\lambda|^2(0, \sin^2\alpha w^v + \cos^2\alpha w + 2\sin\alpha\cos\alpha v \times w - \sin^2\alpha v \times w \times v)\\
=&|\lambda|^2(0, \sin^2\alpha w^v + \cos^2\alpha(w^v + w^\perp) + \sin\theta v \times w^\perp - \sin^2\alpha w^\perp)\\
=&|\lambda|^2(0, (\sin^2\alpha + \cos^2\alpha)w^v + (\cos^2\alpha - \sin^2\alpha)w^\perp + \sin\theta v \times w^\perp)\\
=&|\lambda|^2(0, (w - w^\perp) + \cos\theta w^\perp + \sin\theta v \times w^\perp).
\end{aligned}
$$

Namely, w rotates about v with angle θ.

Quaternion and Rotation

Lemma

The universal covering space of $SO(3)$ is $\mathbb{S}^3$.

Proof.

The set of all unit quaternions $|\lambda|^2 = 1$ is $\mathbb{S}^3$, which is simply connected. Since

$$\bar{\lambda} q \lambda = \overline{(-\lambda)} q (-\lambda),$$

so antipodal points $\pm q$ represent the same rotation, $\mathbb{S}^3$ double covers $SO(3)$. □

The set of the unit quaternions is defined as

$$\text{Spin}(3) := \{q \in \mathbb{H} | q\bar{q} = 1\},$$

which is homemorphic to $\mathbb{S}^3$. $SO(3)$ is homeomorphic to $\mathbb{RP}^3$.

Surface Regular Homotopy

Definition (Surface Regular Homotopy)

Let f and F be two immersions of a given abstract surface M into $\mathbb{R}^3$. We say that f is regular homotopic to F, denoted by $f \sim_r F$, if there exists a continuous family of immersions f_t of M into $\mathbb{R}^3$ such that $f_0 = f$ and $f_1 = F$.

The set of all immersions regularly homotopic to f is denoted by $[f]$, the space of regular homotopy classes of immersions of a given surface M into $\mathbb{R}^3$ bt $\overline{\mathrm{Imm}}(M, \mathbb{R}^3)$.

Regular Immersion

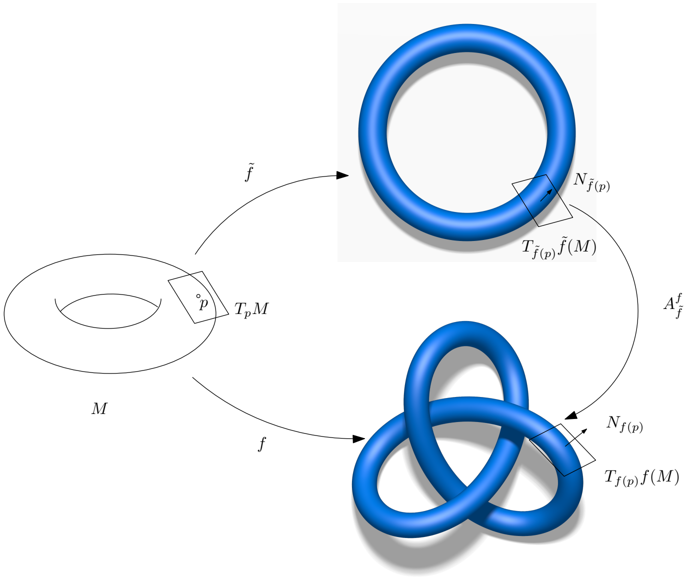
Let M be an oriented surface, $f : M \to \mathbb{R}^3$ is an immersion, then its differential df is a rank two $\mathbb{R}^3$-valued one form on M, df is a section of the bunddle of orientation-preserving rank 2 $\mathbb{R}^3$-valued one forms on M,

$$T^*M \otimes \mathbb{R}^3_+ := \{\tau \in T^*M \otimes \mathbb{R}^3 | \text{rank}(\tau) = 2\}$$

Theorem (Hirsch)

Every continuous section of $T^*M \otimes \mathbb{R}^3_+$ is homotopic to the differential of an immersion of M into $\mathbb{R}^3$.

Regular Immersion



Surface Regular Homotopy

Theorem (James-Thomas)

Let M be an oriented surface. The mapping sending an immersion f to its differential df descends to a bijection

$$\overline{Imm}(M,\mathbb{R}^3) \to [M, SO(3)],$$

where $[M, SO(3)]$ denotes the space of homotopy classes of maps from M into $SO(3)$.

Proof.

Chose a reference immersion $\tilde{f} : M \to \mathbb{R}^3$, denote by $\tilde{N}$ the Gauss map of $\tilde{f}$. Let τ be an orientation preserving rank 2 $\mathbb{R}^3$-valued one form on M, $\tau \in T^*M \otimes \mathbb{R}^3_+$. □

Surface Regular Homotopy

Proof.

τ defines a vector space isomorphism

$$\begin{aligned}\beta_\tau : T_{\pi(\tau)}M \oplus \mathbb{R} &\to \mathbb{R}^3 \\ (X, \rho) &\to \tau(X) + \rho N\end{aligned}$$

Let $A_\tau \in GL(3, \mathbb{R})^+$ be the unique orientation preserving matrix, which makes the diagram commute:

$$\begin{array}{ccc} T_{\pi(\tau)}M \oplus \mathbb{R} & \xrightarrow{\beta_\tau} & \mathbb{R}^3 \\ id \downarrow & & \downarrow A_\tau \\ T_{\pi(d\tilde{f})}M \oplus \mathbb{R} & \xrightarrow{\beta_{d\tilde{f}}} & \mathbb{R}^3 \end{array}$$

□

Surface Regular Homotopy

Proof.

The map

$$A : T^*M \otimes \mathbb{R}^3_+ \to M \times GL(3,\mathbb{R})^+$$
$$\tau \in T^*M \otimes \mathbb{R}^3_+ \to (\pi(\tau), A_\tau)$$

is a bundle map which descends to a map

$$\langle A \rangle : \overline{\text{Imm}}(M,\mathbb{R}^3) \to [M, GL(3,\mathbb{R})^+]$$
$$[f] \to \langle A_{df} \rangle$$

where $\langle \cdot \rangle$ denotes the homotopy class of a map. By Hirsch theorem, the map $\langle A \rangle$ is a bijection. The polar decomposition of $GL(3,\mathbb{R})^+$ implies that $[M, SO(3)] = [M, GL(3,\mathbb{R})^+]$. □

Regular Homotopy

Different regular homotopy classes of immersions of M into $\mathbb{R}^3$ can be distinguished by the twist they assign to topologically nontrivial curves. Let $\gamma : [0, l] \to M$ be a continuous loop. For every continuous $SO(3)$-valued map $A : M \to SO(3)$, let $\alpha : [0, l] \to SO(3)$ be the map induced by the loop γ, $\alpha = A \circ \gamma$. Denote by $\tilde{\alpha}$ an arbitrary lift to Spin(3) of α,

$$A(\gamma(t))V = \overline{\tilde{\alpha}(t)} V \tilde{\alpha}(t), \quad \forall t \in [0, l], V \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H}).$$

Either $\tilde{\alpha}(0)$ and $\tilde{\alpha}(l)$ are the same or antipodal, we obtain

$$\tilde{\alpha}(0)\tilde{\alpha}(l)^{-1} = \pm 1$$

and that $\tilde{\alpha}(0)$ and $\tilde{\alpha}(l)$ depends only on the loop α but not on the particular lift.

Regular Homotopy

Definition (Flip)

We call the number

$$\varphi_A(\gamma) := \tilde{\alpha}(0)\tilde{\alpha}(l)^{-1}$$

the flip of γ with respect to the map A.

- By continuity $\varphi_A(\gamma)$ depends only on the homotopy class $[A]$ of A and the class $[\gamma]$ of the loop γ.
- A continuious map $A: M \to SO(3)$ is homotopically trivial if and only if $\varphi_A(\gamma) = 1$ for every loop γ in M.
- Two $SO(3)$-valued maps A and B are homotopic if and only if $\varphi_A(\gamma) = \varphi_B(\gamma)$ for every continuous loop γ.

Regular Homotopy

Let $\tilde{f} : M \to \mathbb{R}^3$ be a reference immersion. For every immersion $f : M \to \mathbb{R}^3$, left A_{df} be the $GL(3, \mathbb{R})^+$-valued map defined with respect to $\tilde{f}$. Then there exists a $SO(3)$-valued map O_{df} homotopic to A_{df}. By James-Thomas, we have $f \sim_r g \iff O_{df} \sim O_{dg}$.

Definition (Relative Twist)

We define the twist of a continuous loop γ by f relative to $\tilde{f}$ by

$$\tau_f^{\tilde{f}}(\gamma) := \varphi_{O_{df}}(\gamma).$$

Corollary

Two immersions f and g are regular homotopic if and only if

$$\tau_f^{\tilde{f}}(\gamma) = \tau_g^{\tilde{f}}(\gamma)$$

for every loop γ. In particular, f is regularly homotopic to $\tilde{f}$ if and only if $\tau_f^{\tilde{f}}(\gamma) = 1$ for every loop γ.

Regular Homotopy

Corollary

Let M be a surface such that $dim H^1(M,\mathbb{Z}_2)=d$, then $\overline{Imm}(M,\mathbb{R}^3)$ has exactly 2^d elements.

Proof.

Let $f:M\to\mathbb{R}^3$ be an immersion, $\tau_f^{\tilde{f}}:\pi_1(M)\to\{+1,-1\}$ is a homomorphism,

$$\tau_f^{\tilde{f}}(\gamma_1\gamma_2)=\tau_f^{\tilde{f}}(\gamma_1)\tau_f^{\tilde{f}}(\gamma_2),\quad \tau_f^{\tilde{f}}(\gamma^{-1})=\tau_f^{\tilde{f}}(\gamma)^{-1}.$$

which is equivalent to the linear map $\tau_f^{\tilde{f}}:H_1(M,\mathbb{Z}_2)\to\mathbb{Z}_2$, by the mapping $(-1)^k:\mathbb{Z}_2\to\{+1,-1\}$. Suppose $\{a_1,b_1,\dots,a_g,b_g\}$ is the basis of $H_1(M,\mathbb{Z}_2)$, then $\tau_f^{\tilde{f}}$ has the matrix representation

$$\left(\tau_f^{\tilde{f}}(a_1),\tau_f^{\tilde{f}}(b_1),\tau_f^{\tilde{f}}(a_2),\tau_f^{\tilde{f}}(b_2),\cdots,\tau_f^{\tilde{f}}(a_g),\tau_f^{\tilde{f}}(b_g)\right).$$

Immersion Regular Homotopy

Let f be an immersion of an oriented surface M into $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$, the immersion induces a conformal structure J on M.

Let $\gamma : [0,1] \to M$ be a regular closed curve. Define movable frame along γ,

$$\left\{ \frac{df(\gamma')}{|df(\gamma')|}, \frac{df(J\gamma')}{|df(J\gamma')|}, N = \frac{df(\gamma') \times df(J\gamma')}{|df(\gamma') \times df(J\gamma')|} \right\}$$

Let $\lambda : [0, l] \to \mathbb{H}^*$ be a continuous quaternion-valued function such that

$$\begin{aligned} \bar{\lambda} E_1 \lambda &= df(\gamma')/|df(\gamma')| \\ \bar{\lambda} E_2 \lambda &= df(J\gamma')/|df(J\gamma')| \\ \bar{\lambda} E_3 \lambda &= N \end{aligned}$$

where $(E_1, E_2, E_3) = (k, j, -i)$ to be the standard basis in $\mathbb{R}^3$. We call the map λ the lift of the moving frame associated with the curve γ. $\lambda(t)$ is determined uniquely up to a sign.

Immersion Regular Homotopy

Definition (Twist)

The twist of γ with respect to f is defined by

$$\tau_f(\gamma) := \lambda(0)\lambda(l)^{-1}$$

belongs to $\mathbb{Z}_2 = \{+1, -1\}$.

Suppose γ is regular homotopic to γ_s, f_r is regular homotopic to f, then

$$\tau_f(\gamma_s) = \tau_f(\gamma) = \tau_{f_r}(\gamma).$$

Immersion Regular Homotopy

Definition (Twist)

The twist of γ with respect to f is defined by

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Suppose γ is regular homotopic to γ_s, f_r is regular homotopic to f, then

$$\tau_f(\gamma_s) = \tau_f(\gamma) = \tau_{f_r}(\gamma).$$

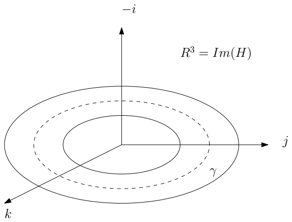
Immersion Regular Homotopy

Problem

Let M be an annulus in the $\mathbb{C}$ $M := \{z|1/2 < |z| < 2\}$. Consider the immersion $f(z) = \mathbf{k}z$, then $df = kdz$. Then $\tau_f(\gamma) = -1$ for every generator γ of $\pi_1(M)$.

Consider the curve $\gamma(t) = -ie^{it}$, $0 \leq t \leq 2\pi$ in M, $\gamma'(t) = e^{it}$, the moving frame is

$$df(\gamma'(t)) = ke^{it}$$
$$df(J\gamma'(t)) = je^{it}$$
$$N(t) = -i$$



The rotatio axis is $-i$, the angle is t, therefore $\lambda(t) = (\cos t/2, -\sin t/2(-i)) = e^{it/2}$. Hence we obtain

$$\lambda(0)\lambda(2\pi)^{-1} = 1 \cdot e^{i\pi} = -1.$$

Immersion Regular Homotopy

Problem

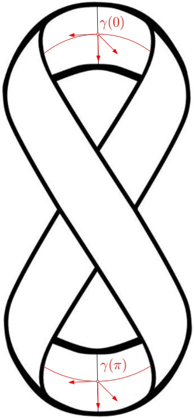
Let M be an annulus in the $\mathbb{C}$ $M := \{z|1/2 < |z| < 2\}$. Consider the immersion f as a ribbon, then $df = kdz$. Then $\tau_f(\gamma) = +1$ for every generator γ of $\pi_1(M)$.

Cut the ribbon along the fiber at 0 and π, each half can be regular homotopic to a straight cylinder, which is regular homotopic to the annulus. Hence

$$\lambda(0)\lambda(\pi)^{-1} = -1 \quad \lambda(\pi)\lambda(2\pi)^{-1} = -1$$

Thus the twist of γ equals

$$\lambda(0)\lambda(2\pi)^{-1} = (\lambda(0)\lambda(\pi)^{-1})(\lambda(\pi)\lambda(2\pi)^{-1}) = +1.$$



Immersion Regular Homotopy

Lemma

Let M be a Riemann surface. For every two conformal immersions $\tilde{f}, f : M \to \mathbb{R}^3 = Im(\mathbb{H})$, for every continuous loop γ in M, we have

$$\tau_f(\gamma) = \tau_{\tilde{f}}(\gamma)\tau_f^{\tilde{f}}(\gamma),$$

Theorem

Let M be a Riemann surface. Two conformal immersions f and $\tilde{f}$ of M into $\mathbb{R}^3$ are regularly homotopic if and only if $\tau_f(\gamma) = \tau_{\tilde{f}}(\gamma)$ for every continuous loop γ in M.

Proof.

From corollary, $f \sim_r \tilde{f}$ if and only if $\tau_f^{\tilde{f}} \equiv 1$. By above lemma, we conclude that $\tau_f^{\tilde{f}} \equiv 1$ if and only if $\tau_f \equiv \tau_{\tilde{f}}$. □